

CANCELLOUS BONE TISSUE REMODELLING: MATHEMATICAL MODELLING

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Abstract. Bone tissue is a heterogeneous, anisotropic material and consists of compact and cancellous bone tissues. The structure of cancellous bone tissue in different parts of the skeleton is not the same. It conforms to Wolff's law: it aims to become optimal for the loading which acts on the corresponding bone; the bone remodelling by means of osteosynthesis and resorption mechanisms. The modern problems of biomechanics demand research of the history of formation of bone structures in the course of time at both physiological and pathological loadings. The mandible is one of the most liable to external and internal bone changes. It is liable to physiological changes that occur during the ontogenesis of the organism. Very often, one has to deal with pathological changes caused by incorrect loading of different regions of bone tissue due to dysfunction of a dentition, a temporomandibular joint and so on. For example, the Popov–Godon's syndrome which connects with tooth loss is accompanied by pathological remodelling of the surrounding bone tissue. Thus, the mathematical modelling of the cancellous bone tissue behavior in the human maxillo-dental system is one of the topical problems of biomechanics and medicine. Structural features of cancellous bone tissue can be described by means of the fabric tensor. This is possible to implement if there is both a constitutive relation which connects the stress tensor, the fabric tensor, and the strain tensor, and kinetic equations which describe the evolution of the fabric tensor and bone density. The phenomenological Cowin's equations are chosen and analyzed in detail as such ones. An initial boundary value problem of the cancellous bone tissue remodelling is stated. The solution of this problem allows us to trace changes in the stress-strain state at the trabecular structure formation according to Wolff's law. The effective numerical algorithm allows us to solve the problem is developed. This algorithm is implemented as a complex of problem-oriented programs. Verification of the model and identification of its parameters are carried out. All numerical calculations are performed using the ANSYS software. Trabecular bone tissue evolution is demonstrated on the set of model examples when the stress–strain state is changed. The results demonstrate different character of influence of changes of loading conditions on the process of structure formation, which follows from Wolff's law.

Key words: cancellous (trabecular) bone tissue, Wolff's law, maxillo-dental system, mandible, fabric tensor, constitutive relation, evolution equation, initial boundary value problem of cancellous bone tissue structure remodelling, ramus of mandible.

INTRODUCTION

Permanently changing loadings of different (physiological and pathological) nature have an influence on the development and functioning of the human skeletal system. Particularly, such changes can happen due to medical influence. It is known [5–7, 33, 38, 48, 63, 64] that such changes of loads have long-term consequences and, at first, affect the bone tissue structure organization. Bone tissue consists of compact (cortical) and cancellous (trabecular) bone according to the position of trabeculae. A trabecula is a structural unit of cancellous bone tissue that forms its architecture.

Cancellous bone tissue is a heterogeneous porous anisotropic structure. Its mechanical properties are defined mostly by its internal architecture and follow the same principles as engineering constructions [7, 44]. Cancellous bone tissue of a living human being is a dynamical structure where metabolism and adaptation processes go on permanently. Adaptation (remodelling) of the cancellous bone tissue structure represents the process of rotation of the trabeculae towards the influence of the predominant force in the studied area. In particular, adaptive changes of the trabecular bone tissue structure can begin under the influence of a new load, which then will affect cortical bone tissue. The remodelling mechanism at the cellular level is implemented by means of resorption of separate old trabeculae by osteoblasts and formation (osteosynthesis) of new ones by osteoclasts.

It is known that adaptation processes occurring in the human body conform to Wolff's law [69]. According to Wolff's law, the following is true concerning living bone tissue. *The bone fits its external shape and internal structure according to the mechanical forces which it should sustain* [6, 7, 33, 35, 36, 44, 48]. Particularly, Wolff's law for bone tissue says that the architecture of the cancellous bone in a local region structurally adapts to a localized stress in the bone tissue [14, 21, 33, 34]. Also, structural adaptation of living cancellous bone has the directional nature; trabeculae are located regularly in accordance with external loading of this bone [6, 7, 35, 48]. Particularly, cancellous bone density increases in regions where loads have been applied and remodelling has begun, and during the remodelling trabeculae line up in accordance with the principal directions of the stress tensor [6, 7, 35, 48].

Though methods for imaging of cancellous bone tissue in vivo currently exist [28, 41, 42], their usage in clinical practice is limited and there is almost no predictive function. Thus, it is necessary to be able to predict the long-time response of the structure to external interference by means of mathematical modelling methods, i.e. to develop the model of remodelling (adaptation) of cancellous bone tissue, taking into account the mutual influence of the stress-strain state and structural changes. Mathematical modelling of the cancellous bone tissue remodelling is still one of the topical problems of biomechanics.

Previously, the author of this paper has described the model of cancellous bone tissue remodelling, taking into account the mutual influence of the stress-strain state and structural changes; this model allows us to predict the long-time response of the structure to external interference by means of mathematical modelling [35, 36, 61]. This model can be used to solve applied problems of the cancellous bone tissue remodelling in different parts of the human skeleton [4, 16–22, 36] for prognostic purposes, particularly, when applied to different parts of the human maxillofacial system [5]. The particular interest is the modelling of the structure in the human mandible.

HUMAN MAXILLODENTAL SYSTEM: THE OBJECT OF THE RESEARCH

The human mandible is an unpaired bone and the only movable one of the facial skeleton. A large number of muscles are attached to it that set it in motion. This feature determines the complexity of the mandible structure and its influence on the development of the facial skeleton and the soft tissue covering the skeleton [1, 2, 38].

Structure of the mandible changes from the birth till extreme old age according to loading of different nature [2]. At the same time, during the development and growth of the mandible, its cancellous bone tissue structure also changes according to its loading and its stress state (Fig. 1).

For objective registration of changes in the bone, it is necessary to explain typical features of a normal structure which are inherent in the jaw in general and follow its different anatomico-physiological laws during each age period. Let us regard these changes in more detail.

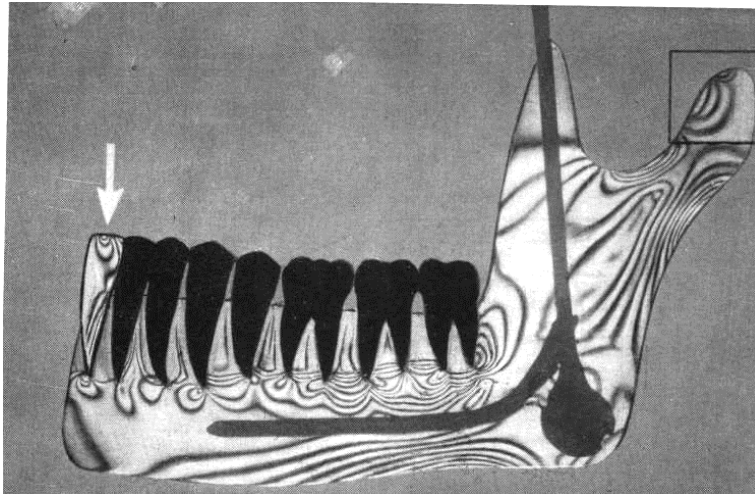


Fig. 1. Stress lines at the loading in the frontal area (the stresses occur in the whole mandible) [37]

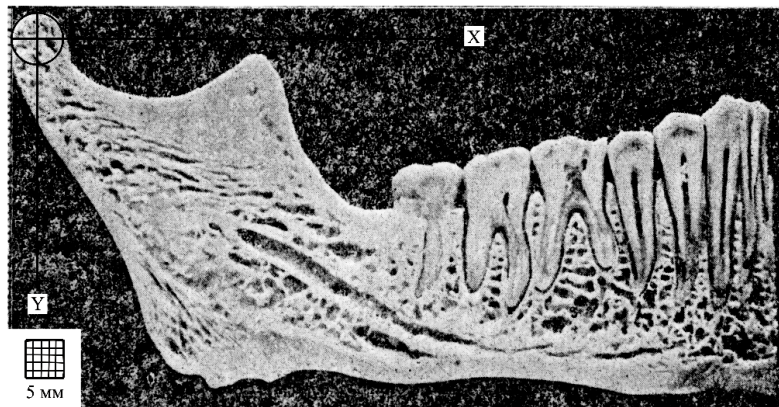


Fig. 2. Cancellous bone tissue of the mandible [37]

It is observed that the body of the mandible has a strongly pronounced structure of a spongy bone tissue (Fig. 2) [37]. The examinations of the trabecular structure have established the regularity of its composition. Bone beams of spongy substance are located along the lines of action of forces under a load. Overall, nine such trajectories are differentiated [64].

Structure of the mandible ramus during all age periods has been studied in papers [2, 64]. The human mandible ramus grows non-uniformly in length and in width [2]. The most intensive growth periods of the mandible ramus in height (length) are marked at the age of 3–4 years and at the age of 9–11 years. The significant growth periods of the mandible ramus in width are observed at the age of the eruption of molars, i.e. at the age of 9, 15, and 18 years [64].

The most increase of bone tissue mass of the ascending mandible ramus takes place from 6 till 15 years. The period from 25 till 45 years inclusively is characterized by the smallest variation amplitude of bone tissue mass of the mandible ramus. From 50 years and up to extreme old age, bone tissue mass of the mandible ramus is decreased. The greatest increase of compact bone tissue mass relative to spongy substance takes place during four periods: from 9 months till 1 year; from 2.5 till 4 years; from 9 till 12 years, and from 15 till 18 years [64].

Formation of spongy substance of the ascending mandible ramus in different ages confirms the data for different types of bone tissue organization in two contiguous triangular regions (ventral and dorsal) of the mandible ramus. There is typical predominance of bow-shaped constructions (in the form of so-called “inverted arches”) in the ventral triangular region (at the mature age); and typical predominance of fan-shaped divergent bone trabeculae in the dorsal one [64].

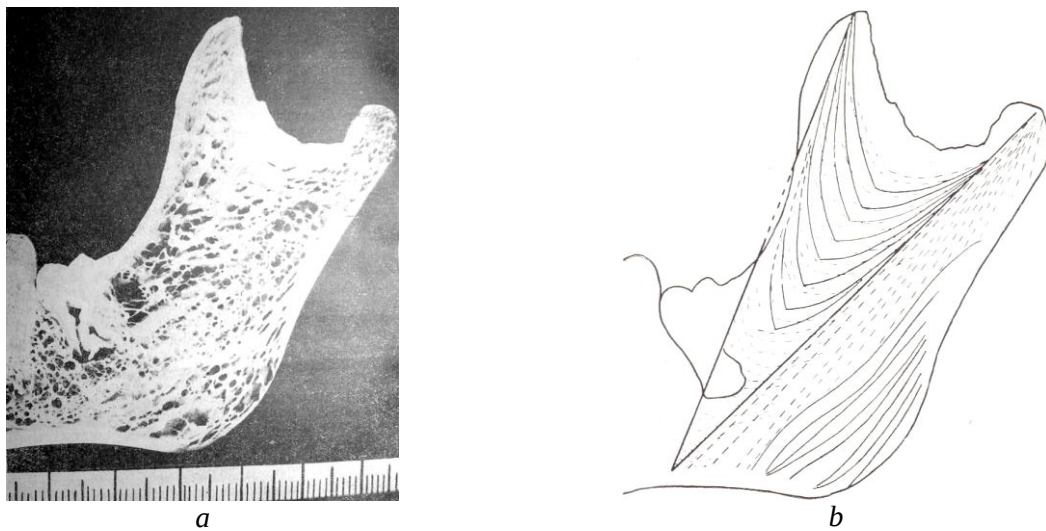


Fig. 3. Structure of the mandible ramus (a) and trajectories of the mandible ramus (b) [2]

Radial, divergent cancellous structures of the mandible ramus can be found already at the time of birth; their amount is rapidly increased by 3 years and then both their number and bearing capacity of constituent units are increased. Bow-shaped bone trabeculae of the ventral region of the mandible ramus are formed later than radial ones. These trabeculae appear for the first time by 2 years in the presence of 16–18 teeth; they develop from 3 till 6 years.

Then, the existing system of “inverted arches” matures, becomes clearer, and progresses in the process of life according to changing life conditions (including the change of the first teeth to the second ones). By 15 years, spongy substance of the ventral triangular region approaches the mature age bone in its structure; by 18 years, it is impossible to distinguish this substance from the one of a 25-year old human (Fig. 3). Dedentition and senile involution show themselves in the greatest modifications of spongy substance in the ventral triangular region of the mandible ramus. Thus, the regressive processes of old age begin in such bone regions which are formed later at human ontogenesis.

It is shown that trabeculae in the alveolar process and mandible ramus line up along the principal stress lines, that is along the principal directions of the stress tensor, and form the arched architecture in the mandible ramus [2, 64]. This statement agrees with Wolff’s law for bone tissue. Thus, it is necessary to have a method of quantitative description of the bone tissue structures formed under the influence of the changing biomechanical pressure in different regions of the dentofacial system. Particularly, it is necessary to consider the internal structure at the statement of boundary value problems on determination of the stress–strain state of a human mandible [31, 34–36, 62–64].

DATA AND METHODS

Since Wolff’s law was formulated, a lot of experimental data and mathematical models which describe this mechanism have been accumulated [4–15, 23, 25, 27, 33, 43, 44, 46, 48, 49, 51, 52, 58, 59, 70].

The process of remodelling of cancellous bone tissue as of an elastic anisotropic body is implemented in this paper; the body has a structure in accordance with the evolutionary relationships which were firstly proposed in the paper [6] and then have been applied in practical calculations in the research series [4, 5, 16, 18–22, 31–36, 48]. The ANSYS software for the finite element analysis and the MATLAB application software are used as a tool for numerical simulation [22].

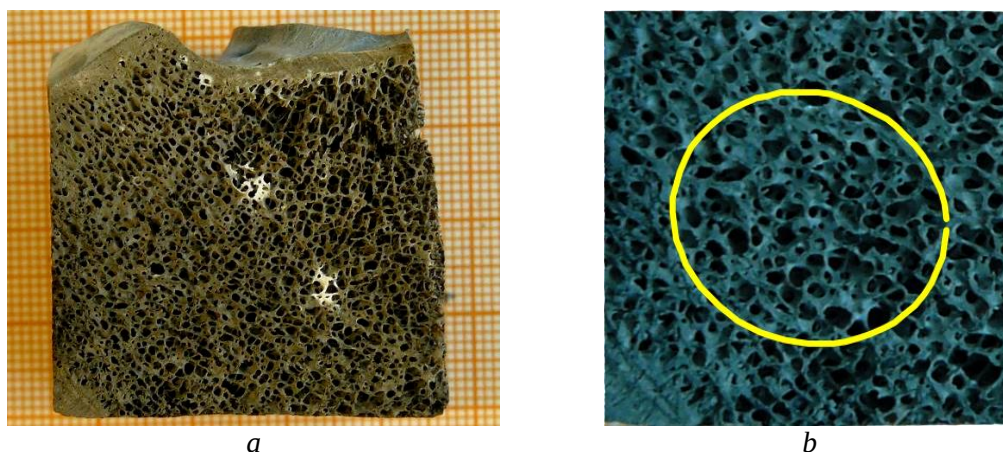


Fig. 4. Typical view of the bone sample after its recovery from the muffle furnace (a) and after its preparation for construction of the fabric tensor with the superimposed fabric ellipse (b) [33]

It is mathematically more convenient to describe a cancellous structure of a region by means of a tensor value at the level of organs and tissues (mesolevel and macrolevel), neglecting the properties of individual trabeculae. It is known that the symmetric positively definite second rank tensor, called the fabric tensor and denoted by $\tilde{\mathbf{H}}$ [7, 33–35, 48, 53, 55, 60–64], is one of the best quantities to describe the local structure of many porous and composite materials [7, 12, 13, 26, 29, 30, 33–35, 44, 46, 48, 49, 53, 55, 58, 60, 64–68].

According to the earlier described method [33, 34, 48, 60, 64], the fabric tensor in cancellous bone tissue allows us to describe the structural anisotropy of bone concisely and in tensorial form, and its principal values allow us to characterize the distribution of material along the principal directions [31, 36, 61, 64]. This tensor allows us to formulate several relations that are able to describe the properties of the material quantitatively, taking into account its structure and adaptation changes of a bone microstructure, for example.

Early the measuring procedure for stereological investigations was analyzed to construct the fabric tensor; properties of the tensor and the connected anisotropy tensor were investigated [33, 60]. Also the procedure of experimental obtaining of three-dimensional cancellous bone tissue and further construction of the fabric tensor was elaborated [33, 60] – see Fig. 4.

It has been found [7, 13, 36, 61] that trabecular orientation in the investigated cancellous bone region at the homeostatic state (the physiological equilibrium state) coincides with principal directions of the stress tensor in the same region. In this case, the stress tensor and the fabric tensor have to be coaxial.

The coincidence of principal axes of the stress tensor $\tilde{\boldsymbol{\sigma}}^0$ and the fabric tensor $\tilde{\mathbf{H}}^0$ occurs in the case of commutativity of their scalar product [7, 13, 48, 56]. That is, it meets the following condition:

$$\tilde{\boldsymbol{\sigma}}^0 \cdot \tilde{\mathbf{H}}^0 = \tilde{\mathbf{H}}^0 \cdot \tilde{\boldsymbol{\sigma}}^0. \quad (1)$$

In previous papers [35, 61], it was demonstrated that there is a constitutive relation that includes the fabric tensor and connects the stress-strain state in the cancellous bone tissue to its structure (trabecular microstructure). It is assumed that bone matrix in a cancellous bone (spongy elastic body) is isotropic [6, 7], and total heterogeneity of cancellous bone is related to the geometry of the anisotropic trabecular bone tissue microstructure. In this case, the anisotropy of the cancellous bone can be described by means of the fabric tensor. At that, the elastic properties of the material will depend on both the cancellous bone porosity and the orientation of the trabeculae.

Thus, on the basis of existing approaches [6–15, 24, 35, 36, 50, 54, 56, 57], it is possible to obtain the constitutive correlation that allows us to describe the stress-strain state of the cancellous bone taking into account its structure:

$$\tilde{\sigma} = (g_1 + g_2 e)(\text{tr} \tilde{\varepsilon}) \tilde{E} + (g_3 + g_4 e) \tilde{\varepsilon} + g_5 (\tilde{\varepsilon} \cdot \tilde{K} + \tilde{K} \cdot \tilde{\varepsilon}) + g_6 (\text{tr}(\tilde{K} \cdot \tilde{\varepsilon}) \tilde{E} + (\text{tr} \tilde{\varepsilon}) \tilde{K}), \quad (2)$$

where $\tilde{K}$ is the deviator of the fabric tensor $\tilde{H}$ normalized as $\text{tr} \tilde{K} = 0$ [24], e is the change in solid volume fraction from a reference value v_0 , and g_1 – g_6 are constants [1, 38] which have the dimension of GPa. These constants were found in the article [59] after a set of experiments on different patterns of human cancellous bones and cattle cancellous bones.

Also, based on Wolff's law [6, 7, 13, 35, 36, 48, 61, 69], a supposed remodelling mechanism was described. This remodelling occurs because of adaptive processes which permanently take place in living bone tissue. According to Wolff's law for cancellous bone tissue [6, 7, 35, 36], evolutionary correlations have been obtained. These correlations are able to describe changes of cancellous bone under the influence of different loadings. Obtained kinetic equations are able to describe the change of the deviator fabric tensor $\tilde{K}$ and the solid volume fraction e . Namely:

$$\begin{aligned} \frac{d\tilde{K}}{dt} = & (h_1 + h_3 e)(\tilde{\varepsilon} - \tilde{\varepsilon}^0) + h_4 \text{tr}(\tilde{\varepsilon} - \tilde{\varepsilon}^0) \tilde{K} + \\ & + h_2 \left((\text{tr}(\tilde{K} \cdot (\tilde{\varepsilon} - \tilde{\varepsilon}^0))) \tilde{E} - \frac{3}{2} (\tilde{K} \cdot (\tilde{\varepsilon} - \tilde{\varepsilon}^0) + (\tilde{\varepsilon} - \tilde{\varepsilon}^0) \cdot \tilde{K}) \right), \end{aligned} \quad (3)$$

and

$$\frac{de}{dt} = (f_1 + f_2 e)(\text{tr} \tilde{\varepsilon} - \text{tr} \tilde{\varepsilon}^0) + f_3 (\text{tr}(\tilde{K} \cdot (\tilde{\varepsilon} - \tilde{\varepsilon}^0))), \quad (4)$$

here h_1 – h_4 и f_1 – f_3 are constants [6, 36], which have the dimension of reciprocal days; these constants are determined empirically in such a way that the bone tissue remodelling occurs in real time. Normally, the cancellous bone adaptation occurs during approximately 160 days [7, 36, 44, 48, 61].

The statement of the initial boundary value problem of trabecular bone tissue remodelling was offered for constitutive relations (2)–(4). This statement can be used to investigate the stress–strain state of the cancellous bone tissue and adaptive processes which occur there.

RESULTS

The numerical algorithm was developed and tested to solve the mathematical model of cancellous bone tissue remodeling presented in this paper. This algorithm was implemented as a complex of problem-oriented programs [4, 5, 31, 36].

Identification and verification of the model parameters have been carried out in the set of computational experiments. The cancellous bone tissue evolution at the changing of the stress–strain state is demonstrated in the set of examples. Further, the results will be briefly described.

With the aim to verify the developed algorithm, we have solved the classical problem [6, 7], which was analyzed in detail in [36, 61]. We have studied a local region of the cancellous bone tissue (Fig. 5) which was in the state of homeostasis during a long-term time period ($t < 0$). At time $t = 0$, the single change of loading conditions occurs which initiates the remodelling of the trabecular microstructure. At that, the new stress state in the considered region doesn't have any changes during another long-term time period ($t > 0$) [36].

Results obtained at the end of 160 days are shown in Fig. 6. The new bone microstructure was obtained as a result of adaptation to the new stress–strain state [36, 61]. At that, the time and the nature of convergence obtained from the solution coincide with the known [6] results.

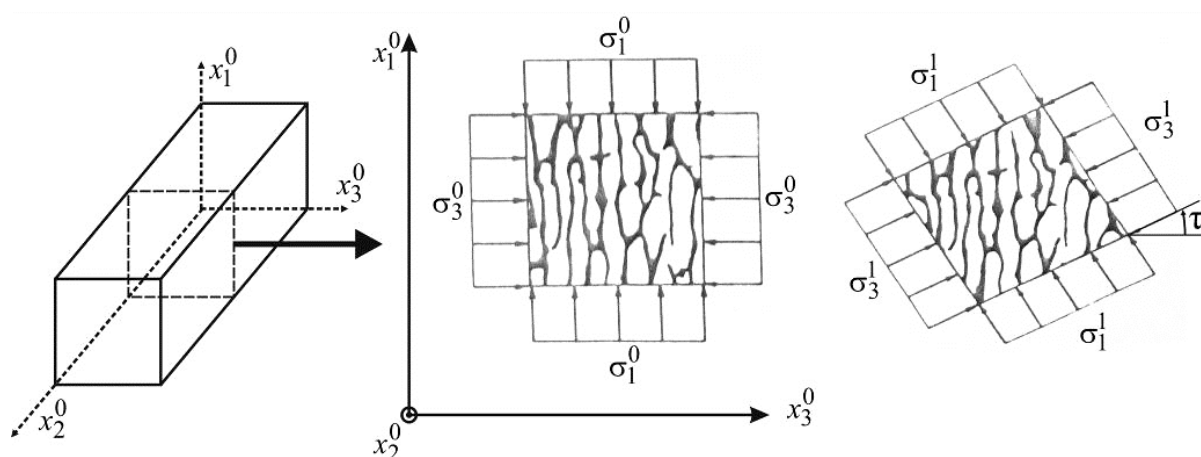


Fig. 5. Illustration of the process of cancellous bone tissue structure remodelling from one physiological equilibrium state to another [36]

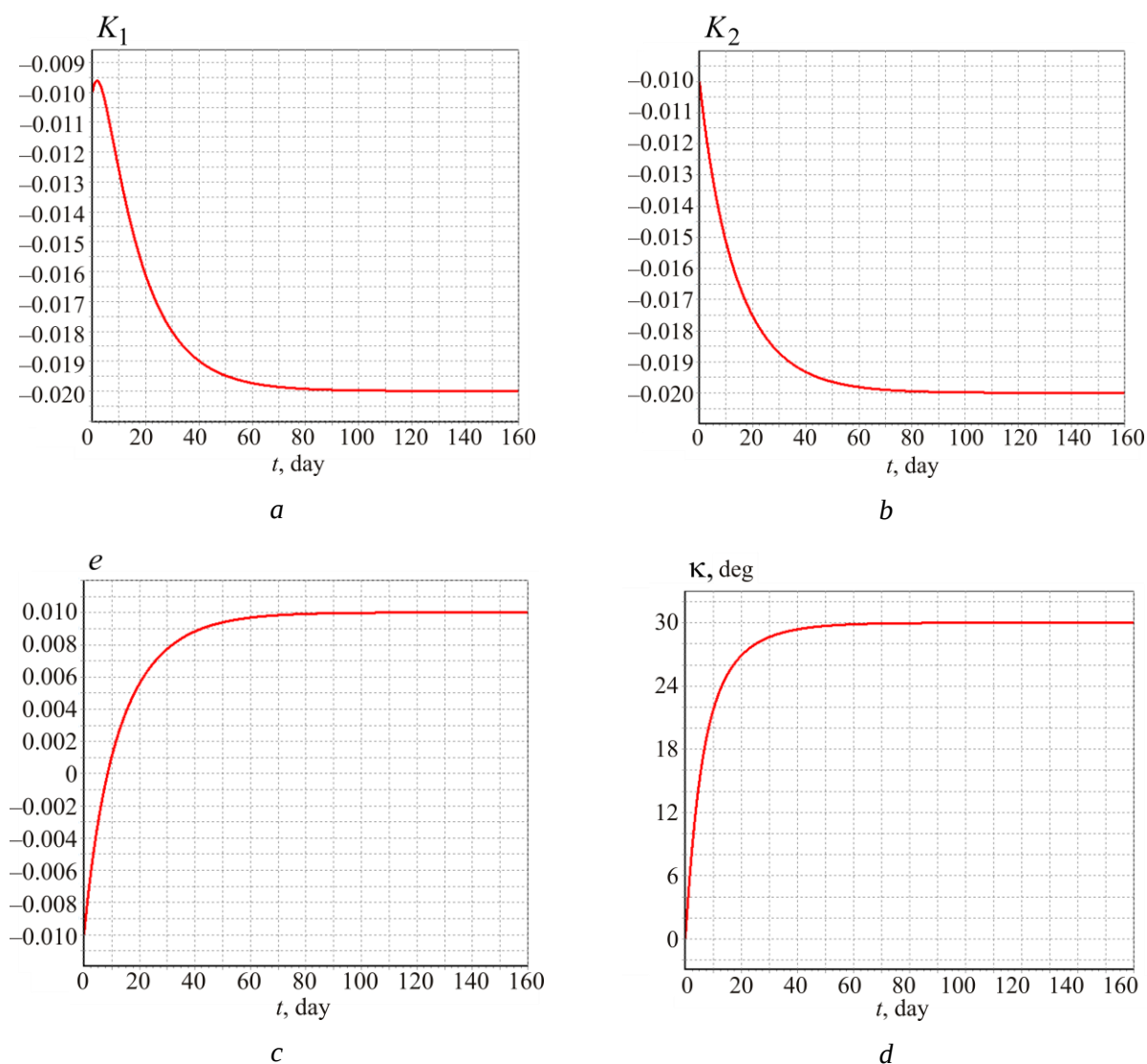


Fig. 6. Evolution of components of the deviator fabric tensor (*a*, *b*), density (*c*), and angle $\kappa(t)$ between the principal axes of the stress tensors and the deviator fabric tensor (*d*) [36]

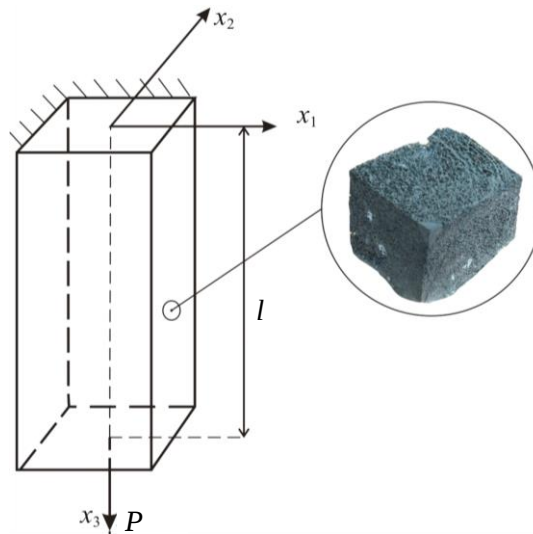


Fig. 7. Tension of a heterogeneous beam under the action of the axial force

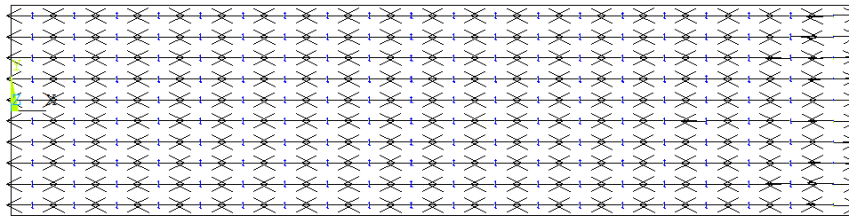


Fig. 8. Principal directions of the deviator fabric tensor after the completion of the remodelling

It is necessary to note that the presented example is complex: the pressure which influences the trabecular microstructure is not initially uniform and later changes its value as well as its direction. This example is difficult to analyze in detail and it is necessary only to compare our results with the same results of the paper [6] and to verify the presented solution. The set of model examples [18–21, 31, 61] which has been considered by the authors and is not presented in this article, demonstrates good efficiency of the model.

Next, the classical problems [19] on tension and bending of a console beam equipped with a structure were considered. The obtained results also prove the working capacity of the model [4, 5, 21, 31].

Using the example of tension of the orthotropic beam [40], the influence of different structure orientations of a material on its stress-strain state was demonstrated. In this problem, the console beam was strained by the axial force $\mathbf{P}$ (see Fig. 7). At that, at each point of the beam the structure was set corresponding to the fabric tensor value for the three-dimensional cancellous bone tissue, obtained in articles [34, 60].

As a result, the initially non-coaxial stress tensor and the deviator fabric tensor became coaxial (see Fig. 8). The new bone microstructure changed as shown in Fig. 9 [4, 19, 31]. The values of displacements, stresses, and strains also coincided with the values determined in [40] for two-dimensional and three-dimensional cases (see Fig. 10–11).

Using the example of bending of a console beam equipped with a structure [40], the influence of different structure orientations of a material on its stress-strain state was demonstrated [21]. Also, it was shown that the results of the numerical simulation coincide qualitatively with the results obtained in the article [27], where a bended console beam takes a specific tongue shape: it becomes thicker in the area of cantilever termination and becomes thinner in the area of bending load application; the principal stress lines form specific trajectories (see Fig. 12). The initially homogeneous beam was bent by the force $\mathbf{F}$ in the problem considered by the author (see Fig. 13).

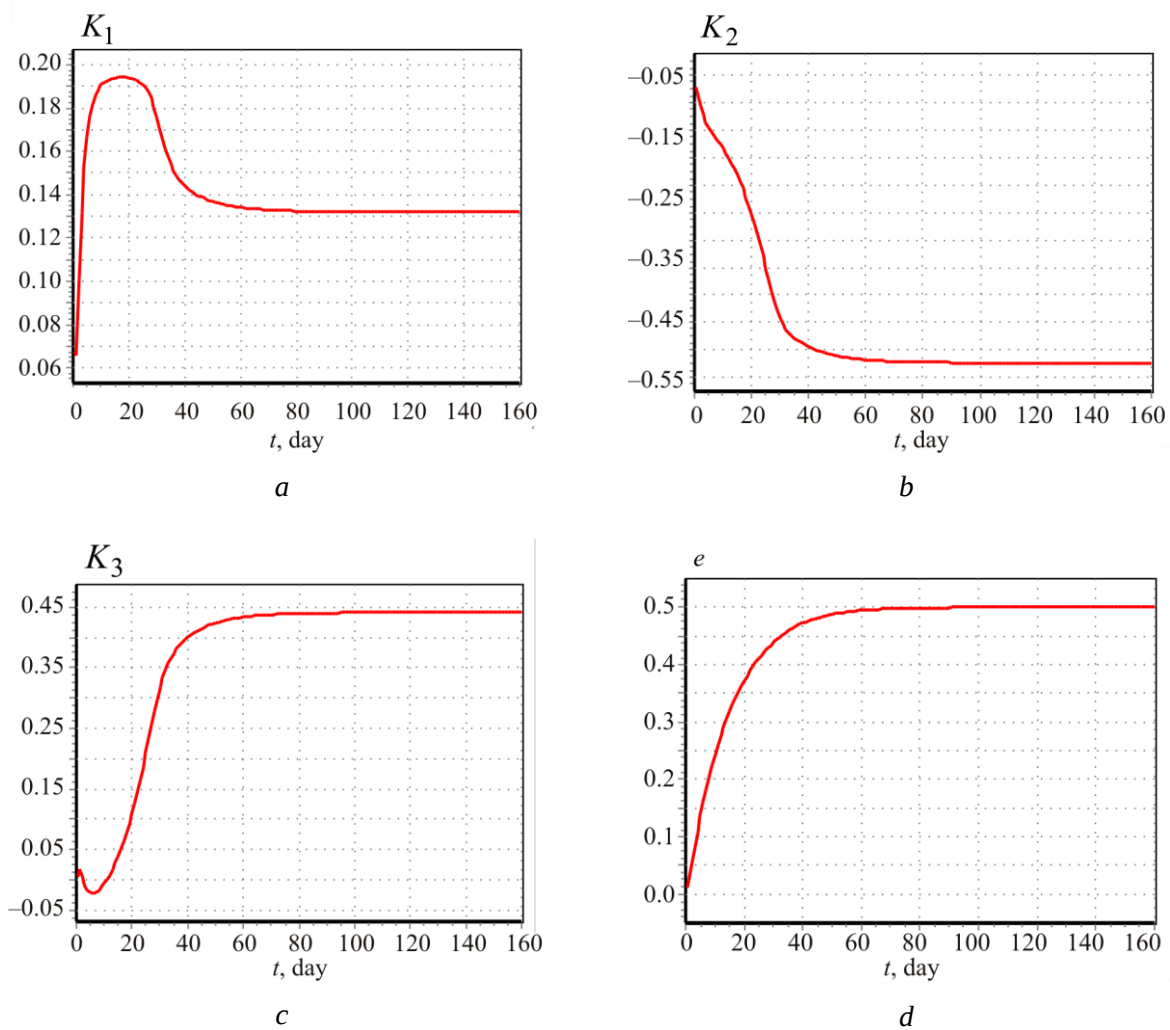


Fig. 9. Evolution of the structure (a component of the deviator fabric tensor $K_1(t)$ (a), $K_2(t)$ (b), and $K_3(t)$ (c)) and the porosity $e(t)$ of the trabecular microstructure (d)

As a result, the initial fabric tensor $\tilde{\mathbf{H}}^0$ corresponding to the homogeneous structure ceased to be spherical. In the process of remodelling, principal axes of the deviator fabric tensor became coaxial with the principal axes of the stress tensor (Fig. 14) [18]. The density of the beam that had initially been uniform increased in the area of cantilever termination and decreased in the area of the load application (see Fig. 15); this fact can be interpreted so that the beam takes the specific shape shown in Fig. 12. The values of displacements, stresses, and strains [4] also coincided with the values determined in [40] (see Fig. 16–17).

THE PROBLEM ON CANCELLOUS BONE TISSUE FORMATION IN THE MANDIBLE RAMUS

Let us consider the problem on the formation and further remodelling of the cancellous bone tissue in the human mandible ramus. The general statement of this problem can be presented as the following system (see Fig. 18).

$$\vec{\nabla} \cdot \tilde{\boldsymbol{\sigma}} = 0, \quad \vec{x} \in V, \quad t \geq 0; \quad (5)$$

$$\tilde{\boldsymbol{\sigma}} = \tilde{\tilde{\mathbf{C}}}(\tilde{\mathbf{K}}, e) \cdot \tilde{\boldsymbol{\varepsilon}}, \quad \vec{x} \in V \cup S, \quad t \geq 0; \quad (6)$$

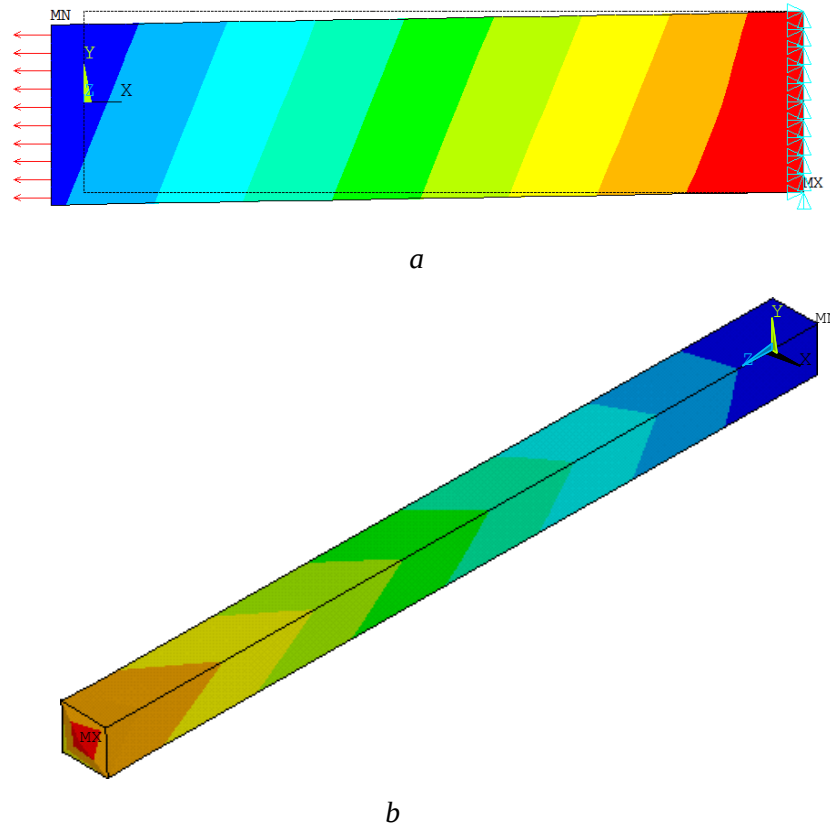


Fig. 10. Isoclines of displacements in a heterogeneous beam after the completion of the remodelling for a two-dimensional (a) and a three-dimensional (b) case

$$\frac{d\tilde{K}}{dt} = \tilde{f}_1(\tilde{\varepsilon}, \tilde{K}, e), \quad \text{tr} \tilde{K} = 0, \quad \bar{x} \in V, \quad t \geq 0; \quad (7)$$

$$\frac{de}{dt} = f_2(\tilde{\varepsilon}, \tilde{K}, e), \quad \bar{x} \in V, \quad t \geq 0; \quad (8)$$

$$\tilde{\varepsilon} = \frac{1}{2}(\bar{\nabla} \bar{u} + \bar{u} \bar{\nabla}), \quad \bar{x} \in V \cup S, \quad t \geq 0; \quad (9)$$

$$\bar{n} \cdot \bar{\sigma}|_{S_{\sigma k}} = \bar{P}_k, \quad t \geq 0, \quad k = 1, 2, 3; \quad (10)$$

$$\bar{n} \cdot \bar{\sigma}|_{S_{\sigma 4}} = 0, \quad t \geq 0; \quad (11)$$

$$\bar{u}|_{S_u} = 0, \quad t \geq 0; \quad (12)$$

$$t = 0: \quad \tilde{K} = \tilde{K}^0, \quad e = e^0, \quad \bar{x} \in V; \quad (13)$$

$$t = 0: \quad \bar{P}_k = \bar{P}_k^0, \quad \bar{x} \in S_{\sigma k}, \quad k = 1, 2, 3; \quad \bar{u} = 0, \quad \bar{x} \in S_u. \quad (14)$$

At the initial time, the structure of the cancellous bone tissue in the mandible ramus was

considered homogeneous and was set by the spherical fabric tensor, i.e. $\tilde{K}^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

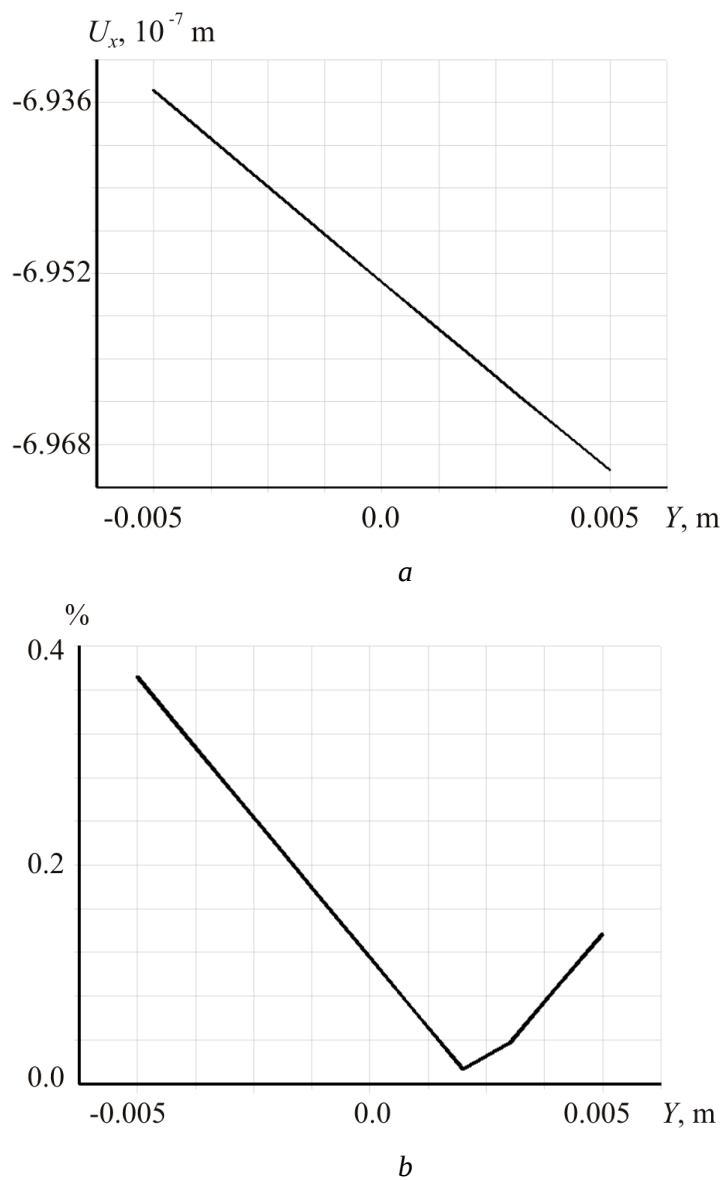


Fig. 11. Displacements of midline points (a) and relative error (b) of the numerical solution in comparison with the analytical one [20]

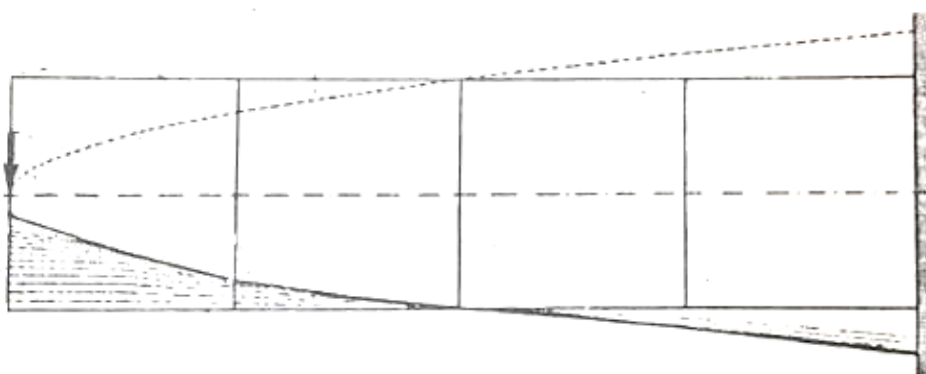


Fig. 12. Adaptation of the shape of a console beam loaded by a bending force [27]

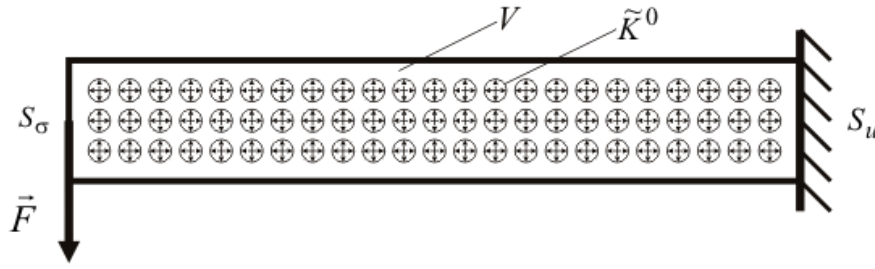


Fig. 13. Bending of a console beam equipped with a homogeneous structure $\tilde{H}^0$ (or $\tilde{K}^0$) at the initial time

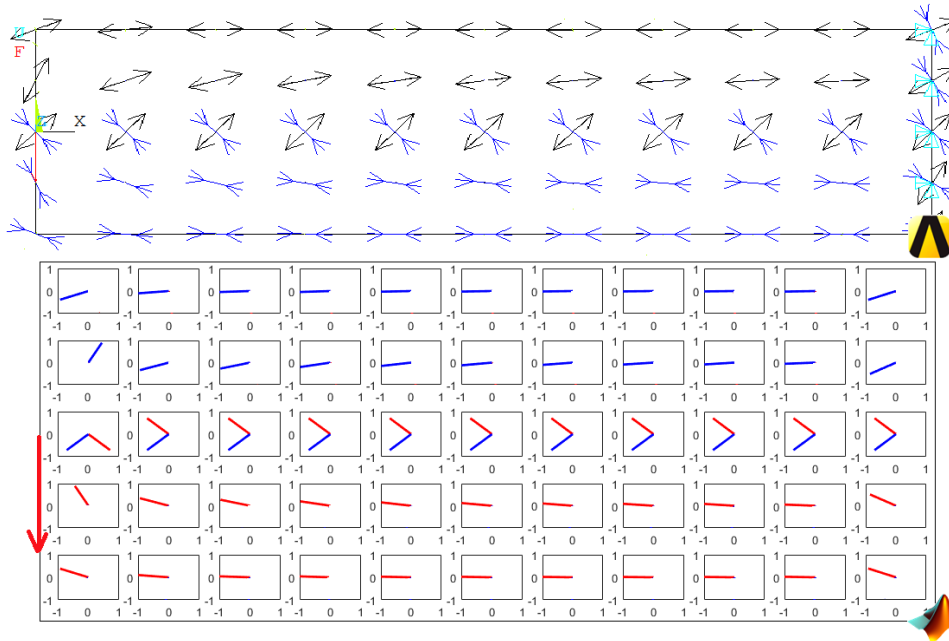


Fig. 14. Comparison of directions of the stress tensor and the deviator fabric tensor principle axes after the completion of the remodelling

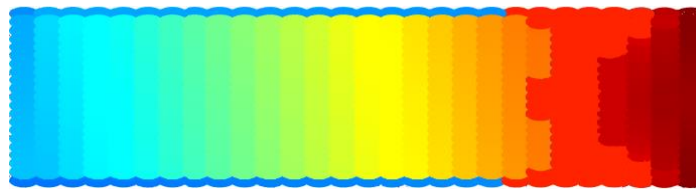


Fig. 15. Change in density (porosity) of a beam after the completion of the remodelling: ■ marks more dense zones, ■ marks less dense zones

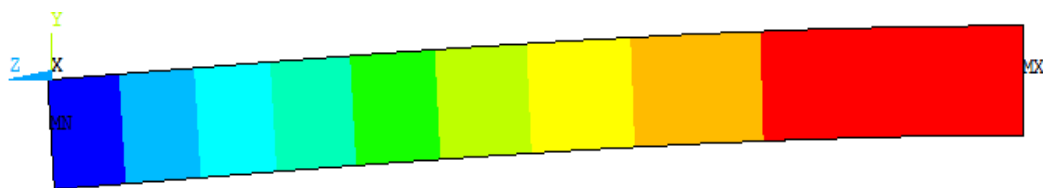


Fig. 16. Isoclines of displacements in a heterogeneous beam after the completion of the remodelling

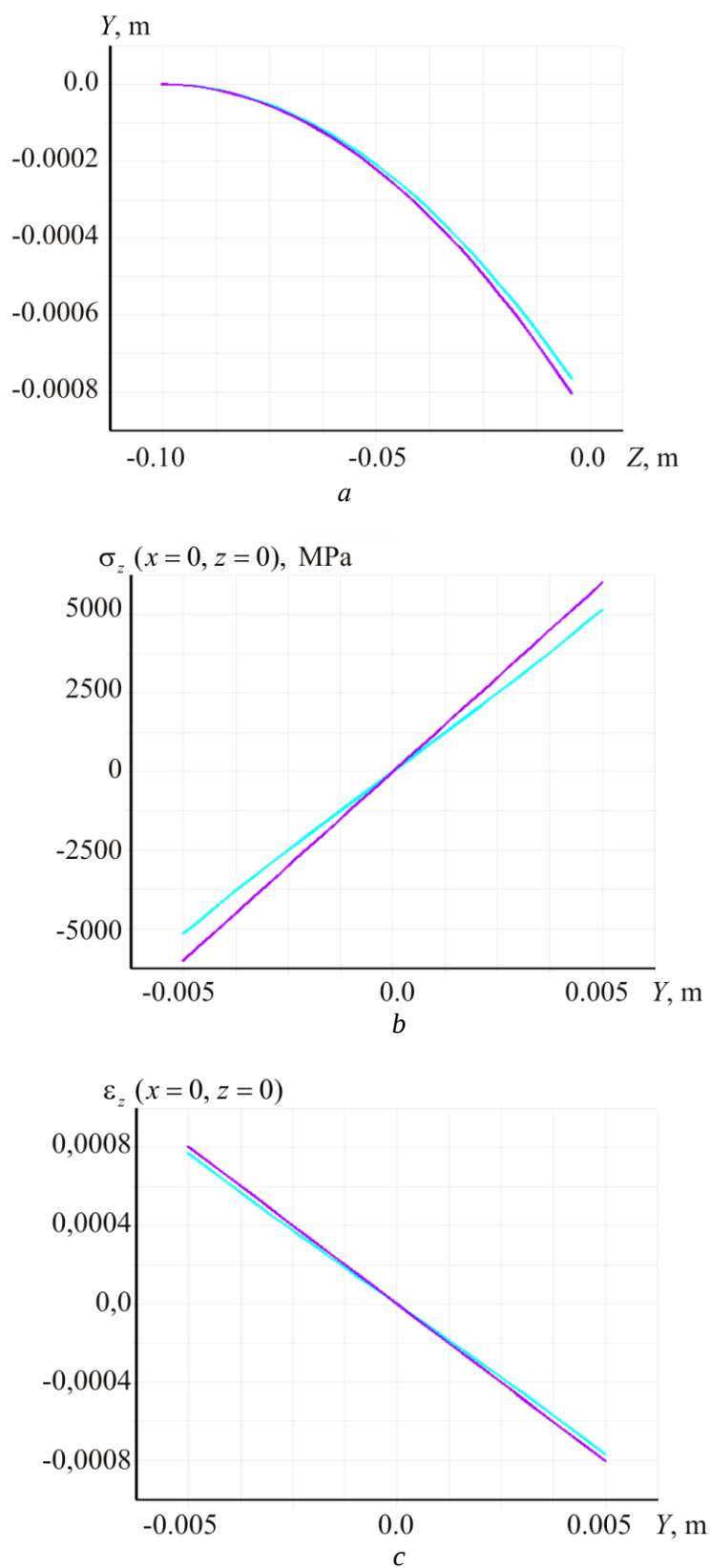


Fig. 17. Comparison of analytical [21] and numerical solutions: displacement of points of the middle line U_y (a), stresses σ_z (b), and strains ε_z (c) in the section perpendicular to the middle line at $z = l / 2$

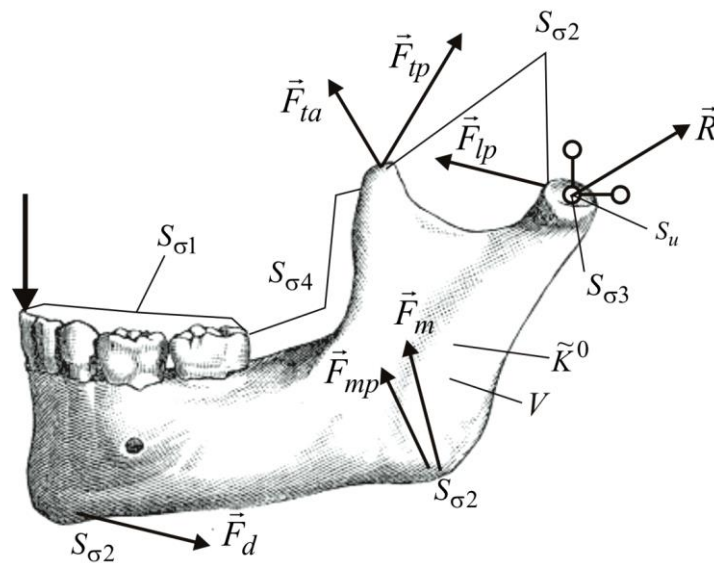


Fig. 18. Statement of the problem on the remodelling of the cancellous bone tissue in the human mandible ramus

Such structure can correspond to the situation when there are no loadings influencing the mandible, that is, it is the mandible of a newborn. Further, when loadings are applied, the structure of the mandible has to take a specific shape (see Fig. 3).

It was previously shown [48, 64] that the fabric tensor can describe the structure in the mandible ramus with a sufficient degree of accuracy; at that, the principal axes of the fabric tensor coincided with the trajectories of trabeculae obtained in the paper [2] and with the stress trajectories obtained in the article [45] (see Fig. 19).

The stress–strain state was calculated with the use of the finite element method in ANSYS [5, 22], the structure parameters (the bone porosity and the deviator fabric tensor) were determined in MATLAB. A two-dimensional finite element model of the mandible was constructed, the mandible dimensions were determined from the article [2], the properties of the bone matrix were found in the paper [39], and the coordinates of the application points of the masticatory muscles and values of their loadings on the bone were taken from the articles [32, 47]. To determine the finite element mesh, we have used the two-dimensional finite element Plane182, which supports the specification of anisotropic properties. The numerical integration of the kinetic equations was performed by the fourth-order Runge–Kutta method.

We have considered different versions of loading application at the initial time. It is necessary to note that the version of the area without taking into account the dentition (the load is applied directly to the mandible between the estimated location of the second and the first premolars) allowed us to demonstrate the initial stress state most clearly – (see Fig. 20).

It can be seen that the obtained image of stress intensity in the mandible ramus (see Fig. 20, *a*) coincides qualitatively with the stress trajectories obtained by the photoelasticity method [45] (see Fig. 1 and 19, *a* and *b*). Also, the obtained results coincide with the results presented in the article [47] for the same area without taking into account the parameters of the structure (see Fig. 20, *b*).

Another illustrative and convenient method for subsequent calculations of the stress state is a plot of the principal directions of the stress tensor, which show how trabeculae are oriented inside the bone tissue (see Fig. 21).

The porosity obtained at the end of 160 days is shown in Fig. 22. The results of the bone density distribution obtained for the mandible geometry with teeth and without them are qualitatively the same for the considered loading conditions. A similarity is seen in the porosity distribution in comparison with Fig. 3, *a*.

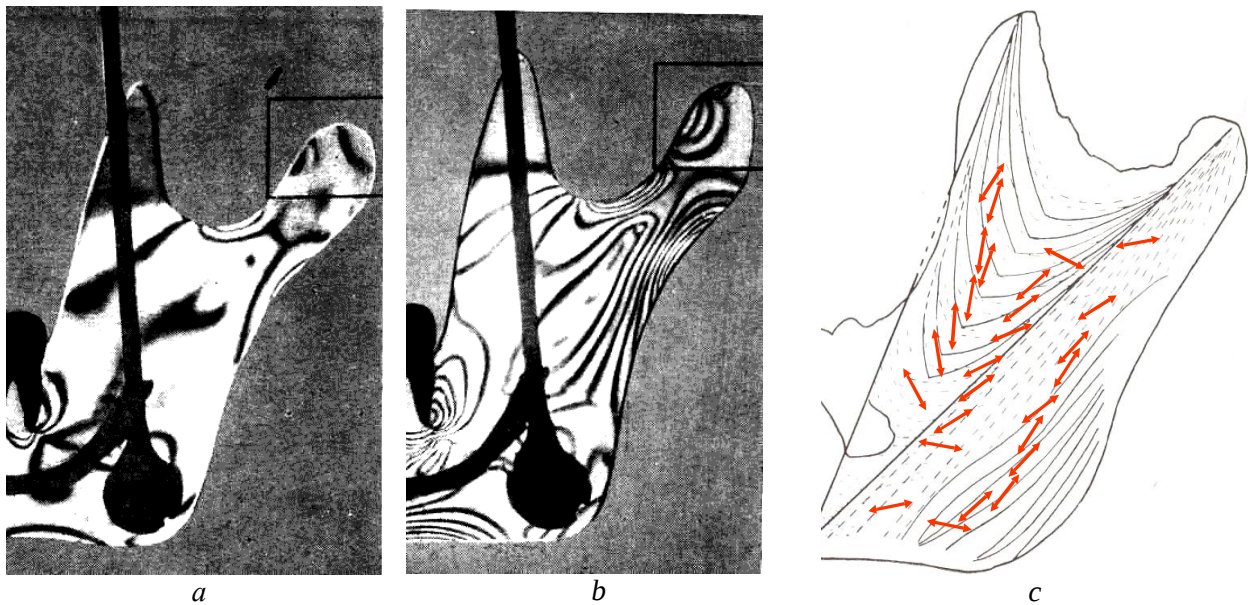


Fig. 19. Comparison of the stress trajectories obtained by means of the photoelasticity method [45] (*a* and *b*), the trajectories of the mandible ramus [2], and the principal directions of the structure tensor (*c*) [64]

As for the trabecular distribution in the mandible ramus, computational experiments have demonstrated that the allocated 160 days are not enough for the principal axes of the deviator fabric tensor to become coaxial with the principal axes of the stress tensor. Apparently, the structure in the mandible ramus needs more time to adapt to the physiological stress state. It corresponds to the known facts: the complete trabecular structure forms only by the age of 16–18 years. The determination of the model parameters and time of the remodelling is the subject of our further study.

CONCLUSIONS

The model of cancellous bone tissue remodelling was studied in this paper. We took into account the constitutive correlation and the kinetic equations of the phenomenological theory [34] which describe changes in the trabecular architecture at the mesolevel using the fabric tensor. The statement of the problem on cancellous bone tissue structure remodelling was stated and its working capacity was demonstrated on the set of model examples.

The bone tissue remodelling in the ramus of mandible has also been considered. The obtained results are quite similar to the stress and structure trajectories which are known from the literature. Thus, the presented model represents regularities of bone tissue formation in different parts of the human skeleton; the model can be used, for example, to describe the response of maxillofacial system to changing biomechanical pressure.

Particularly, the modelling of bone tissue behavior in the region of implantation is interesting, for example, in Popov–Godon’s syndrome [63, 64] (Fig. 23) or some kind of a functional trauma [22] with the further substitution of dentition imperfections. One of the main problems of contemporary dentistry is the atrophy of jawbones in particular adentia or full secondary adentia. Loss of secondary teeth and lack of timely restorative treatment are basic etiological factors of evolution of “pathological” type of atrophy in the alveolar process.

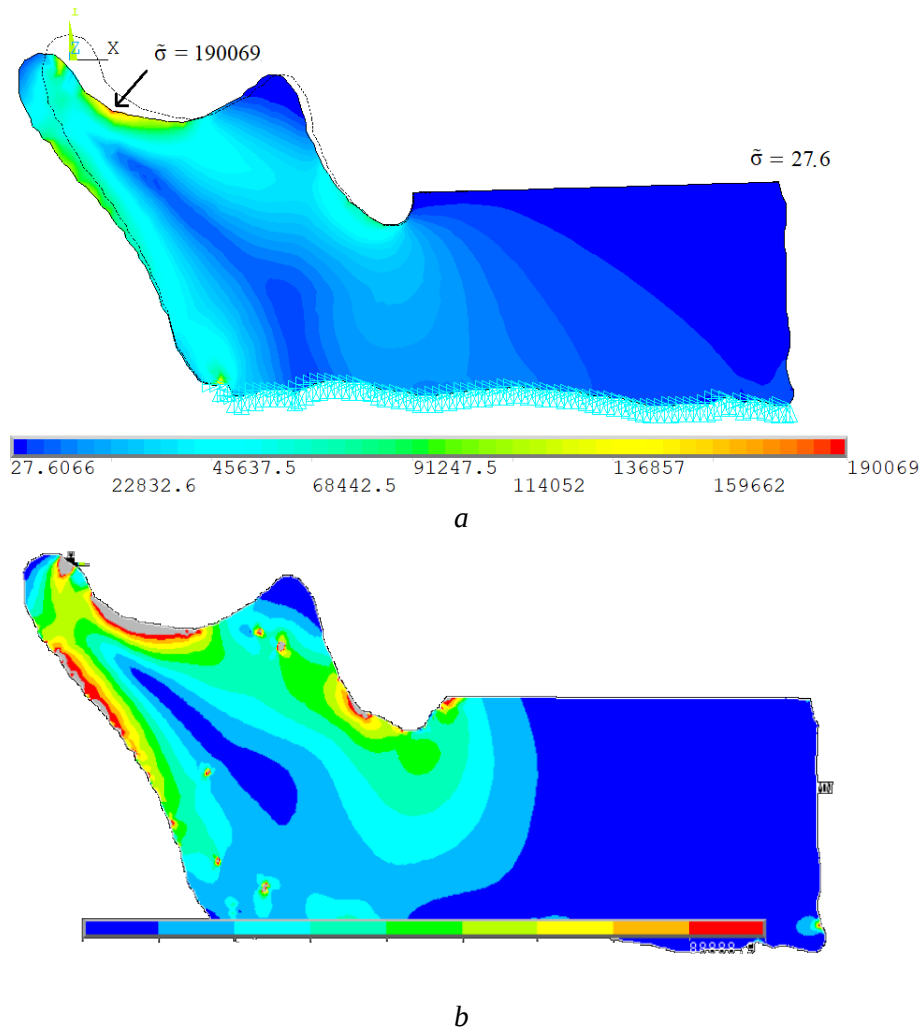


Fig. 20. Von Mises stress (a) and its comparison with the article (b) [47]

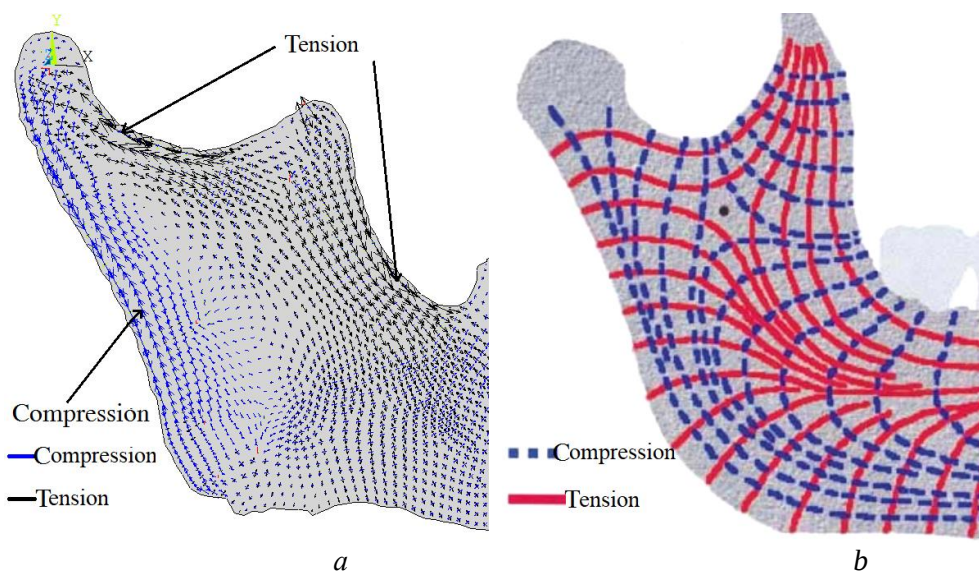


Fig. 21. Simplified representation of the distribution of stresses: principal directions calculated in ANSYS [5] (a) and trajectories observed on average in a mandible [39] (b)

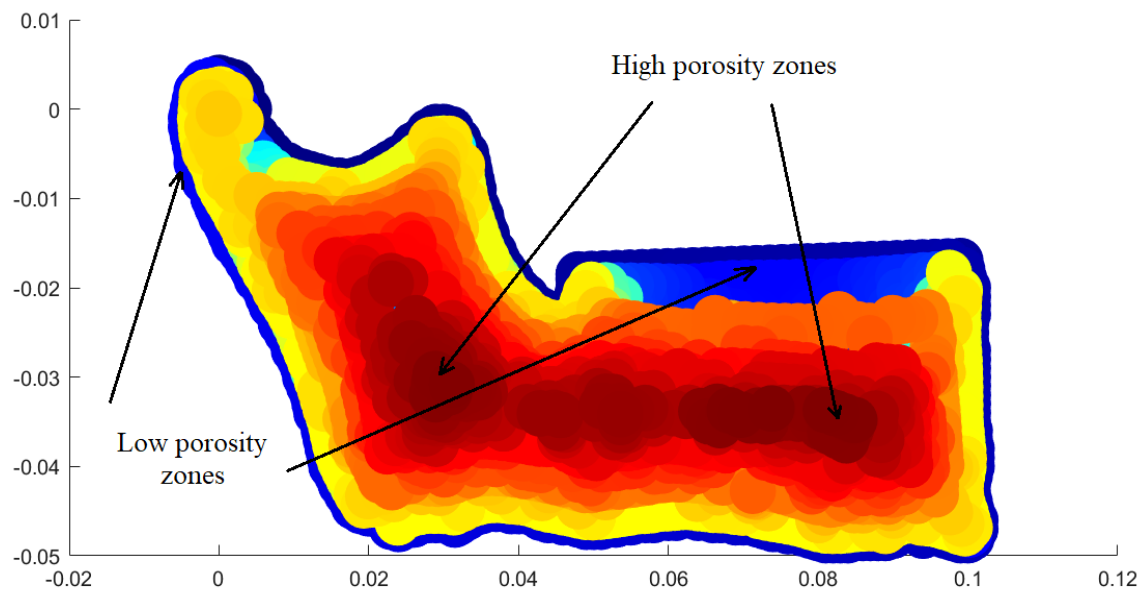


Fig. 22. Distribution of porosity after 160 days

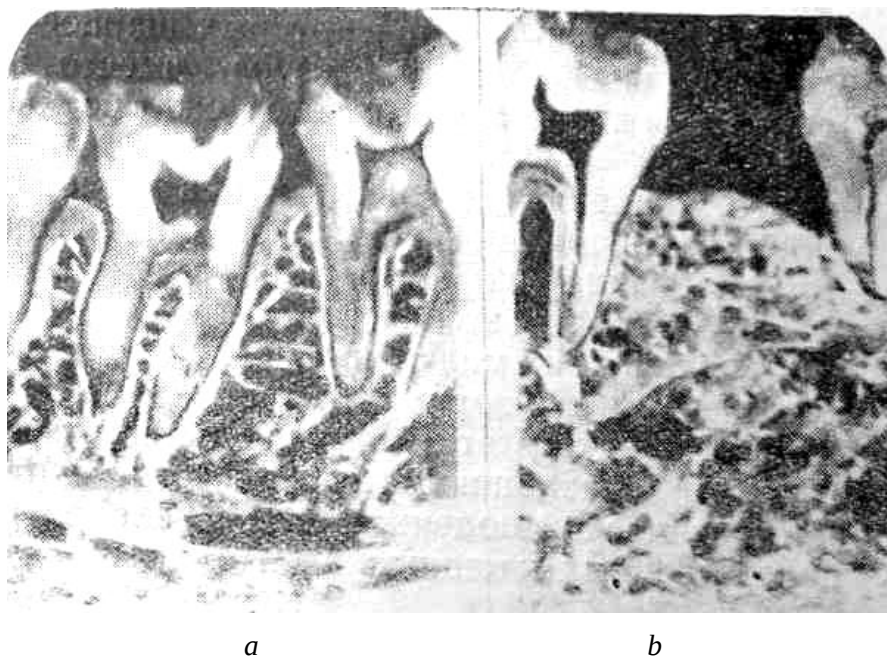


Fig. 23. A normal spongy substance structure of a mandible of a male at the age of 25 years (preparation) (a) and a spongy substance structure of a mandible of a male at the same age with Popov-Godon's syndrome (preparation) (b) [17]

It is known that when a tooth is lost, cancellous bone tissue gets the characteristic diffuse structure, and trabeculae are arranged chaotically; it can be represented as the spherical fabric tensor (see Fig. 23 and 24, a). Further, when the dentition defect is replaced with an implant, trabeculae line up perpendicular to the generatrix of the tooth root (or implant pin), see Fig. 24, b. The same can be said on the orthodontic movement of a tooth: a homogeneous spongy structure should obtain a shape known in orthodontics (Fig. 24).

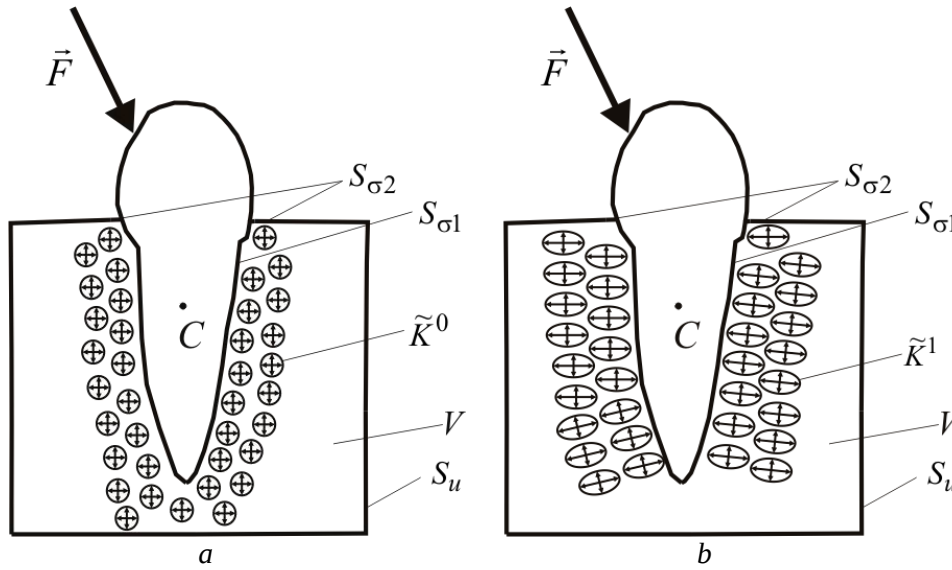


Fig. 24. Statement of the problem on bone tissue remodelling at the restoration of dentition in Popov-Godon's syndrome

In this case, the boundary and initial conditions will take the following form:

$$\vec{n} \cdot \vec{\sigma}|_{S_{\sigma 1}} = \vec{F}_n, \quad t \geq 0;$$

$$\vec{n} \cdot \vec{\sigma}|_{S_{\sigma 2}} = 0, \quad t \geq 0;$$

$$\vec{u}|_{S_u} = 0, \quad t \geq 0;$$

$$t = 0: \quad \tilde{K} = \tilde{K}^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e = e^0, \quad \vec{x} \in V;$$

$$t = 0: \quad \vec{F} = \vec{F}^0, \quad \vec{x} \in S_{\sigma 1}; \quad \vec{u} = 0, \quad \vec{x} \in S_u.$$

Solution of such problems allows us to plan the treatment of maxillofacial system diseases and to understand better the mechanisms of its functioning in the future. It becomes possible to predict the results of remote medical operations in an individual approach to each patient.

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