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DIMENSIONLESS MODELS OF DEFORMATION OF STIFFENED SHELL STRUCTURES

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ABSTRACT

This paper suggests a variant of dimensionless parameters for a wide range of shell structures. Dimensionless parameters have been used for shallow shells of rectangular planform for a long time. There is no universal form of dimensionless relations for shells of a general type, since the Lamé parameters are different for each type of shells, as far as their values and dimensions are concerned. Therefore, the work shows the dimensionless relations for strains, stresses, forces, moments and the functional of total potential energy of deformation. These relations consider geometrical nonlinearity, transverse shears, material orthotropy, as well as the introduction of ribs according to the structural anisotropy method with their shear and torsional rigidity taken into account. The authors show a further approach to solving strength and stability problems with respect to different types of shells in dimensionless parameters. Some methods intended to solve nonlinear strength problems are not quite correct, when dimensional parameters are used (for example, the methods based on the best parameter continuation method). In dimensionless parameters, all calculation are beyond doubt, as far as their correctness is concerned. The article also provides the calculations of some shell structures in dimensionless and dimensional parameters and shows their consistency. The introduction of dimensionless parameters in the course of calculation of such structures allows to obtain more comprehensive information on the stress and strain state of shells and detect strain features for a whole range of such shells. Also, this approach is suitable for optimizing the choice of structural parameters. The authors analyze some differences in critical loads that were obtained in the dimensionless and dimensional solutions of the problem.

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1. Introduction

Shell structures are widely used in various industries, and the study of the process of their deformation is an important task [1-4]. During operation, the shells may be subjected to various loads, and therefore it is necessary to analyze their strength, stability and non-linear vibrations. Shell structures belong to a large class of geometric forms; in particular, most often designed shallow bicurved shells, cylindrical, conical, spherical, toroidal shells and their panels. In addition, they can be reinforced with ribs to increase stiffness.

The introduction of dimensionless parameters allows to obtain the stress and strain state of a whole range of such shells by one calculation, thereby choosing the most rational dimensional parameters of shells, as well as shell material for isotropic shells.

When calculating shallow isotropic shells of rectangular planform, dimensionless parameters were used in the works [5-12]. For rotational shells in the axially symmetric arrangement, the dimensionless parameters were used in the work [13], which is a special case of the parameters described in this work.

The works [14-31] use different dimensionless parameters for each specific type of shells and problems to be solved. However, it is reasonable to develop universal dimensionless parameters that are suitable for different types of shells regardless of a problem to be solved.

2. Problem Setting

The purpose of this work is to develop the dimensionless parameters for different types of shells, calculate strength and stability of shells in dimensionless and dimensional parameters and analyze results to be obtained.

3. Theory and Methods

3.1. Main Relations of the Mathematical Model

3.1.1. Relations for Unstiffened Shells

The mathematical model of deformation of shells consists of three relation groups:

– geometrical relations that connect strains and displacements;

– physical relations that connect stresses and strains;

– the functional of the total potential energy of shell deformation, the minimum condition of which allows to derive equilibrium equations.

The geometrical relations in the shell middle surface with geometrical nonlinearity taken into account take the known form [32]

$$\varepsilon_x=\frac{1}{A}\frac{\partial U}{\partial x}+\frac{1}{AB}V\frac{\partial A}{\partial y}-k_xW+\frac{1}{2}\theta_1^2,$$

$$\varepsilon_y=\frac{1}{B}\frac{\partial V}{\partial y}+\frac{1}{AB}U\frac{\partial B}{\partial x}-k_yW+\frac{1}{2}\theta_2^2,$$

$$\gamma_{xy}=\frac{1}{A}\frac{\partial V}{\partial x}+\frac{1}{B}\frac{\partial U}{\partial y}-\frac{1}{AB}U\frac{\partial A}{\partial y}-\frac{1}{AB}V\frac{\partial B}{\partial x}+\theta_1\theta_2, \tag{1}$$

$$\theta_1=-\left(\frac{1}{A}\frac{\partial W}{\partial x}+k_xU\right),\quad \theta_2=-\left(\frac{1}{B}\frac{\partial W}{\partial y}+k_yV\right),$$

where $\varepsilon_x,\varepsilon_y$ are the elongation strains along the x, y coordinates of the middle surface; γ_{xy} are the shear strains in the xOy plane; $k_x=\frac{1}{R_1}$, $k_y=\frac{1}{R_2}$ are the main curvatures of the shell along the x and y axes; R_1,R_2 are the main radii of the curvature along the x and y axes; A,B are the Lamé parameters that characterize the shell geometry; $U=U(x,y),V=V(x,y),W=W(x,y)$ are the displacements of points of the shell middle surface along the x,y,z axes.

We will consider transverse shears, in this case

$$\gamma_{xz}=kf(z)\left[\Psi_x-\theta_1\right],\quad \gamma_{yz}=kf(z)\left[\Psi_y-\theta_2\right]. \tag{2}$$

Here $f(z)$ is the function that characterizes the distribution of the stresses τ_{xz} and τ_{yz} along the shell thickness; $\Psi_x=\Psi_x(x,y),\Psi_y=\Psi_y(x,y)$ are the angles of rotation of the normal in the xOz and yOz planes respectively; $k=\frac{5}{6}$.

For unstiffened shells [15]

$$f(z)=6\left(\frac{1}{4}-\frac{z^2}{h^2}\right),$$

where h is the thickness of the shell.

The physical relations under linear elastic deformation for the orthotropic material under plane stress will have the form [33]

$$\sigma_x=\frac{E_1}{1-\mu_{12}\mu_{21}}\left[\varepsilon_x+\mu_{21}\varepsilon_y+z\left(\chi_1+\mu_{21}\chi_2\right)\right];$$

$$\sigma_y=\frac{E_2}{1-\mu_{12}\mu_{21}}\left[\varepsilon_y+\mu_{12}\varepsilon_x+z\left(\chi_2+\mu_{12}\chi_1\right)\right]; \tag{3}$$

$$\tau_{xy}=G_{12}\left[\gamma_{xy}+2z\chi_{12}\right];$$

$$\tau_{xz}=G_{13}\gamma_{xz};\ \tau_{yz}=G_{23}\gamma_{yz}.$$

Here E_1, E_2, μ_{12}, μ_{21}, G_{12}, G_{13}, G_{23} are the mechanical properties of the material and χ_1,χ_2,χ_{12} are the curvature change and torsion functions:

$$\chi_1=\frac{1}{A}\frac{\partial\Psi_x}{\partial x}+\frac{1}{AB}\frac{\partial A}{\partial y}\Psi_y;\quad \chi_2=\frac{1}{B}\frac{\partial\Psi_y}{\partial y}+\frac{1}{AB}\frac{\partial B}{\partial x}\Psi_x,$$

$$2\chi_{12}=\frac{1}{A}\frac{\partial\Psi_y}{\partial x}+\frac{1}{B}\frac{\partial\Psi_x}{\partial y}-\frac{1}{AB}\left(\frac{\partial A}{\partial y}\Psi_x+\frac{\partial B}{\partial x}\Psi_y\right).$$

The functional of the total potential energy of deformation for the unstiffened orthotropic shell can be represented as follows:

$$E_p=\frac{E_1 h}{2(1-\mu_{12}\mu_{21})}\int_{a_1}^{a}\int_0^b\Big\{\varepsilon_x^2+\bar{G}_2\varepsilon_y^2+2\mu_{21}\varepsilon_x\varepsilon_y+\bar{G}_{12}\gamma_{xy}^2+$$
$$+\bar{G}_{13}k(\Psi_x-\theta_1)^2+\bar{G}_{23}k(\Psi_y-\theta_2)^2+$$
$$+\frac{h^2}{12}\left(\chi_1^2+\bar{G}_2\chi_2^2+2\mu_{21}\chi_1\chi_2+4\bar{G}_{12}\chi_{12}^2\right)-$$
$$-2\frac{q(1-\mu_{12}\mu_{21})}{E_1 h}W\Big\}ABdxdy. \qquad (4)$$

Taking into account that for the orthotropic material $E_1\mu_{21}=E_2\mu_{12}$, here we introduce the notations

$$\bar{G}_2=\frac{E_2}{E_1},\ \bar{G}_{12}=\frac{G_{12}(1-\mu_{12}\mu_{21})}{E_1},$$
$$\bar{G}_{13}=\frac{G_{13}(1-\mu_{12}\mu_{21})}{E_1},\ \bar{G}_{23}=\frac{G_{23}(1-\mu_{12}\mu_{21})}{E_1}.$$

The expressions of forces and moments for unstiffened orthotropic shells have the form [34]:

$$N_x=\frac{E_1}{1-\mu_{12}\mu_{21}}\left[h\left(\varepsilon_x+\mu_{21}\varepsilon_y\right)\right],$$

$$N_y=\frac{E_2}{1-\mu_{12}\mu_{21}}\left[h\left(\varepsilon_y+\mu_{12}\varepsilon_x\right)\right],\ N_{xy}=N_{yx}=G_{12}h\gamma_{xy},$$

$$M_x=\frac{E_1}{1-\mu_{12}\mu_{21}}\left[\left(\frac{h^3}{12}\right)\left(\chi_1+\mu_{21}\chi_2\right)\right],$$

$$M_y=\frac{E_2}{1-\mu_{12}\mu_{21}}\left[\left(\frac{h^3}{12}\right)\left(\chi_2+\mu_{12}\chi_1\right)\right], \qquad (5)$$

$$M_{xy}=M_{yx}=G_{12}\left[2\left(\frac{h^3}{12}\right)\chi_{12}\right],\ Q_x=G_{13}kh(\Psi_x-\theta_1),$$

$$Q_y=G_{23}kh(\Psi_y-\theta_2),$$

where N_x, N_y, N_{xy}, N_{yx} are the normal forces in the *x*, *y* direction and shear forces in the *xOy* plane respectively; M_x, M_y, M_{xy}, M_{yx} are the bending moments in the *x*, *y* direction and torques; Q_x, Q_y are the transverse forces in the *xOz* and *yOz* planes.

3.1.2. Relations for Reinforced Shells

In order to consider the reinforcement of a structure with stiffeners at a sufficiently great number of ribs, we will use the structural anisotropy method [35]. The essence of this method consists in reducing the shell of the discrete variable thickness to the shell equal with respect to rigidity to the shell of constant thickness and allows to consider such material factors as shear and torsional rigidity of ribs. Also, this method can be modified in order to calculate shells weakened with cutouts.

For the shell reinforced with stiffeners $f(z)$ takes the form [35]

$$f(z)=-\frac{6}{(h+H)^2}\left(z+\frac{h}{2}\right)\left(z-\frac{h}{2}-H\right).$$

Here

$$H(x,y)=\sum_{j=1}^{m}h^j\bar{\delta}(x-x_j)+\sum_{i=1}^{n}h^i\bar{\delta}(y-y_i)-$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{m}h^{ij}\bar{\delta}(x-x_j)\bar{\delta}(y-y_i),$$

where *h* is height; the indices *i* and *j* indicate a rib's number located parallel to the x and y axes respectively; *n*, *m* is the quantity of the ribs; $h^{ij}=\min\{h^i,h^j\}$; $\bar{\delta}(x-x_j)$ and $\bar{\delta}(y-y_i)$ represent differences of two unit functions $\bar{\delta}(x-x_j)=U(x-a_j)-U(x-b_j)$; $\bar{\delta}(y-y_i)=U(y-c_i)-U(y-d_i)$, where $a_j=x_j-r_j/2$, $b_j=x_j+r_j/2$, $c_i=y_i-r_i/2$, $d_i=y_i+r_i/2$. Therefore, r is the width of the rib; the thickness of the whole structure is equal to *h* + *H*.

The functional of the total potential energy of deformation will have the following form [34, 36]

$$E_p=\frac{E_1}{2(1-\mu_{12}\mu_{21})}\int_{a_1}^{a}\int_0^b\Big\{(h+F_x)\varepsilon_x^2+\bar{G}_2(h+F_y)\varepsilon_y^2+$$
$$+\mu_{21}(2h+F_x+F_y)\varepsilon_x\varepsilon_y+\frac{1}{2}\bar{G}_{12}(2h+F_x+F_y)\gamma_{xy}^2+$$
$$+\bar{G}_{13}k(h+F_x)(\Psi_x-\theta_1)^2+\bar{G}_{23}k(h+F_y)(\Psi_y-\theta_2)^2+$$
$$+2S_x\varepsilon_x\chi_1+\mu_{21}(S_x+S_y)\varepsilon_x\chi_2+\mu_{21}(S_x+S_y)\varepsilon_y\chi_1+$$
$$+2\bar{G}_2S_y\varepsilon_y\chi_2+2\bar{G}_{12}(S_x+S_y)\gamma_{xy}\chi_{12}+\left(\frac{h^3}{12}+J_x\right)\chi_1^2+$$
$$+\bar{G}_2\left(\frac{h^3}{12}+J_y\right)\chi_2^2+\mu_{21}\left(\frac{h^3}{6}+J_x+J_y\right)\chi_1\chi_2+$$
$$+2\bar{G}_{12}\left(\frac{h^3}{6}+J_x+J_y\right)\chi_{12}^2-2\frac{q(1-\mu_{12}\mu_{21})}{E_1}W\Big\}ABdxdy. \qquad (6)$$

Where F_x, F_y, S_x, S_y, J_x, J_y is the area of the cross or section of the rib; the area attributable to the unit of cross length; a static moment and an inertia moment of this section that have the form [34]

$$F_x=\sum_{i=1}^{n}\frac{h^ir_i}{\tilde{b}}+\sum_{j=1}^{m}\left(\frac{h^jr_j}{\tilde{a}}-\sum_{i=1}^{n}\frac{h^{ij}r_ir_j}{\tilde{a}\tilde{b}}\right)\frac{r_j}{\tilde{a}};$$

$$F_y = \sum_{j=1}^{m} \frac{h^j r_j}{\tilde{a}} + \sum_{i=1}^{n} \left(\frac{h^i r_i}{\tilde{b}} - \sum_{j=1}^{m} \frac{h^{ij} r_i r_j}{\tilde{a}\tilde{b}} \right) \frac{r_i}{\tilde{b}};$$

$$S_x = \sum_{i=1}^{n} \frac{S^i r_i}{\tilde{b}} + \sum_{j=1}^{m} \left(\frac{S^j r_j}{\tilde{a}} - \sum_{i=1}^{n} \frac{S^{ij} r_i r_j}{\tilde{a}\tilde{b}} \right) \frac{r_j}{\tilde{a}};$$

$$S_y = \sum_{j=1}^{m} \frac{S^j r_j}{\tilde{a}} + \sum_{i=1}^{n} \left(\frac{S^i r_i}{\tilde{b}} - \sum_{j=1}^{m} \frac{S^{ij} r_i r_j}{\tilde{a}\tilde{b}} \right) \frac{r_i}{\tilde{b}}; \qquad (7)$$

$$J_x = \sum_{i=1}^{n} \frac{J^i r_i}{\tilde{b}} + \sum_{j=1}^{m} \left(\frac{J^j r_j}{\tilde{a}} - \sum_{i=1}^{n} \frac{J^{ij} r_i r_j}{\tilde{a}\tilde{b}} \right) \frac{r_j}{\tilde{a}};$$

$$J_y = \sum_{j=1}^{m} \frac{J^j r_j}{\tilde{a}} + \sum_{i=1}^{n} \left(\frac{J^i r_i}{\tilde{b}} - \sum_{j=1}^{m} \frac{J^{ij} r_i r_j}{\tilde{a}\tilde{b}} \right) \frac{r_i}{\tilde{b}},$$

where

$$S^i = \frac{h^i\left(h+h^i\right)}{2}, \quad S^j = \frac{h^j\left(h+h^j\right)}{2}, \quad S^{ij} = \frac{h^{ij}\left(h+h^{ij}\right)}{2},$$

$$J^i = 0,25h^2h^i + 0,5h\left(h^i\right)^2 + \frac{1}{3}\left(h^i\right)^3,$$

$$J^j = 0,25h^2h^j + 0,5h\left(h^j\right)^2 + \frac{1}{3}\left(h^j\right)^3,$$

$$J^{ij} = 0,25h^2h^{ij} + 0,5h\left(h^{ij}\right)^2 + \frac{1}{3}\left(h^{ij}\right)^3.$$

Here the $\tilde{a}, \tilde{b}$ variables allow the stiffeners to have the constant width and are defined as $\tilde{a} = aA, \ \tilde{b} = bB,$ where $B(x) = B\left(\frac{a+a_1}{2}\right)$

The expressions of forces and moments for reinforced orthotropic shells have the form [34, 36]:

$$N_x = \frac{E_1}{1-\mu_{12}\mu_{21}} \left[\left(h+F_x\right)\left(\varepsilon_x + \mu_{21}\varepsilon_y\right) + S_x\left(\chi_1 + \mu_{21}\chi_2\right) \right],$$

$$N_y = \frac{E_2}{1-\mu_{12}\mu_{21}} \left[\left(h+F_y\right)\left(\varepsilon_y + \mu_{12}\varepsilon_x\right) + S_y\left(\chi_2 + \mu_{12}\chi_1\right) \right],$$

$$N_{xy} = G_{12}\left[\left(h+F_y\right)\gamma_{xy} + 2S_y\chi_{12} \right], \qquad (8)$$

$$N_{yx} = G_{12}\left[\left(h+F_x\right)\gamma_{xy} + 2S_x\chi_{12} \right],$$

$$M_x = \frac{E_1}{1-\mu_{12}\mu_{21}} \left[S_x\left(\varepsilon_x + \mu_{21}\varepsilon_y\right) + \left(\frac{h^3}{12} + J_x\right)\left(\chi_1 + \mu_{21}\chi_2\right) \right],$$

$$M_y = \frac{E_2}{1-\mu_{12}\mu_{21}} \left[S_y\left(\varepsilon_y + \mu_{12}\varepsilon_x\right) + \left(\frac{h^3}{12} + J_y\right)\left(\chi_2 + \mu_{12}\chi_1\right) \right],$$

$$M_{xy} = G_{12}\left[S_y\gamma_{xy} + 2\left(\frac{h^3}{12} + J_y\right)\chi_{12} \right],$$

$$M_{yx} = G_{12}\left[S_x\gamma_{xy} + 2\left(\frac{h^3}{12} + J_x\right)\chi_{12} \right],$$

$$Q_x = G_{13}k\left(h+F_x\right)\left(\Psi_x - \theta_1\right), \ Q_y = G_{23}k\left(h+F_y\right)\left(\Psi_y - \theta_2\right)$$

3.2. Dimensionless Parameters for Unstiffened Orthotropic Shells

Let us introduce dimensionless parameters

$$\xi = \frac{x-a_1}{a-a_1}, \quad \eta = \frac{y}{b}, \quad \bar{\lambda} = \frac{aA}{bB}, \quad k_\xi = hk_x, \quad k_\eta = hk_y,$$

$$\bar{U} = \frac{aUA}{h^2}, \bar{V} = \frac{bVB}{h^2}, \bar{W} = \frac{W}{h},$$

$$\bar{\Psi}_x = \frac{\Psi_x aA}{h}, \bar{\Psi}_y = \frac{\Psi_y bB}{h}, \qquad (9)$$

$$\bar{\sigma}_x = \frac{\sigma_x\left(1-\mu_{12}\mu_{21}\right)a^2A^2}{E_1h^2},$$

$$\bar{\sigma}_y = \frac{\sigma_y\left(1-\mu_{12}\mu_{21}\right)a^2A^2}{E_2h^2}, \quad \bar{\tau}_{xy} = \frac{\tau_{xy}a^2A^2}{G_{12}h^2},$$

$$\bar{P} = \frac{a^4A^4q}{h^4E_1}, \quad \bar{A} = \frac{aA}{h}, \quad \bar{B} = \frac{bB}{h}, \quad \bar{z} = \frac{z}{h},$$

where a, b are linear sizes of the shell in the x and y directions according, $\bar{\lambda}$ is the dimensionless coefficient; $\xi, \eta, \bar{z}$ are new (dimensionless) coordinates of the shell.

The Lamé parameters A, B of the shell surface play an important role in the formulation of dimensionless parameters. The values of the Lamé parameters for cylindrical, spherical, conical, toroidal shells are given in Table 1.

Table 1

Lamé parameters for main types of rotational shells

Type of shell	A	B	R_1, m	R_2, m	a	b
Shallow, with rectangular planform	1	1	r_1	r_2	meters	meters
Cylindrical	1	r	∞	r	meters	radians
Spherical	r	$r \cdot \sin x$	r	r	radians	radians
Conical	1	$x \cdot \sin\theta$	∞	$x \cdot \tan\theta$	meters	radians
Toroidal	r	$d + r \cdot \sin x$	r	$\frac{d + r \cdot \sin x}{\sin x}$	radians	radians

Here R_1, R_2 are the main radii of the curvature of the shell along the *x*, *y* axes; *d* is the displacement of the generating sector from the rotational axis for the toroidal shell; θ is the conicity angle of the conical shell.

As Table 1 shows, the parameter *b* is dimensionless; if *a* is dimensional, then *A* is dimensionless and vice versa, i.e. *aA* is the dimensional quantity (meters); *B* = *B*(*x*) is the dimensional quantity (meters), i.e. *bB* is the dimensional quantity. Therefore, if we introduce the Lamé dimensionless parameters as follows $\bar{A} = \frac{aA}{h}, \bar{B} = \frac{bB}{h}$, then they will be dimensionless for all the types of shells under consideration.

The calculation of the stress and strain state and stability of shell structures in dimensionless parameters allows to obtain the critical load for a whole range of homothetic shells (Fig. 1, 2). Herewith deflections, forces and moments for such different shells are found by scaling the obtained dimensionless values of these quantities.

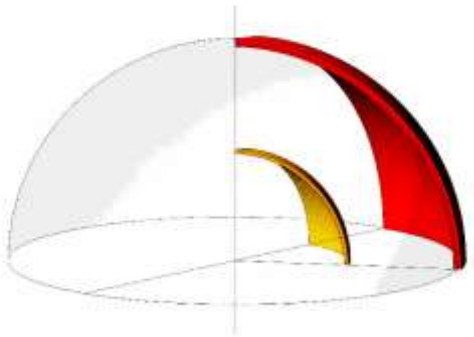

Fig. 1. Homothetic spherical shells

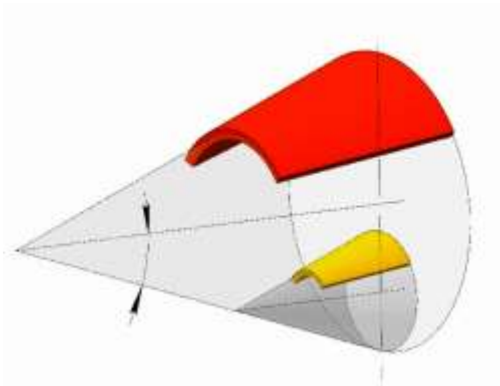

Fig. 2. Homothetic conical shells

Let us pass to the dimensionless parameters (9) in the relations (1), (2) after preparing the expressions $\frac{1}{AB}\frac{\partial B}{\partial x}, \theta_1, \theta_2$ in the dimensionless parameters:

$$\frac{1}{AB}\frac{\partial B}{\partial x} = \frac{1}{AB}\frac{h}{ab}\frac{\partial \bar{B}}{\partial \xi} = \frac{h}{AaBb}\frac{\partial \bar{B}}{\partial \xi};$$

$$\theta_1 = -\left(\frac{h}{aA}\frac{\partial \bar{W}}{\partial \xi} + \frac{1}{h}k_\xi \frac{h^2}{aA}\bar{U}\right) = -\frac{h}{aA}\left(\frac{\partial \bar{W}}{\partial \xi} + k_\xi \bar{U}\right) = \frac{h}{aA}\bar{\theta}_1;$$

$$\theta_2 = -\left(\frac{h}{bB}\frac{\partial \bar{W}}{\partial \eta} + \frac{1}{h}k_\eta \frac{h^2}{bB}\bar{V}\right) =$$

$$= -\frac{h}{aA}\bar{\lambda}\left(\frac{\partial \bar{W}}{\partial \eta} + k_\eta \bar{V}\right) = \frac{h}{aA}\bar{\lambda}\bar{\theta}_2;$$

$$\varepsilon_x = \frac{1}{A}\frac{\partial U}{\partial x} - k_x W + \frac{1}{2}\theta_1^2 =$$

$$= \frac{h^2}{a^2A^2}\left(\frac{\partial \bar{U}}{\partial \xi} - k_\xi \bar{A}^2\bar{W} + \frac{1}{2}\bar{\theta}_1^2\right) = \frac{h^2}{a^2A^2}\bar{\varepsilon}_x;$$

$$\varepsilon_y = \frac{1}{B}\frac{\partial V}{\partial y} + \frac{1}{AB}\frac{\partial B}{\partial x}U - k_y W + \frac{1}{2}\theta_2^2 =$$

$$= \frac{h^2}{a^2A^2}\left(\bar{\lambda}^2\frac{\partial \bar{V}}{\partial \eta} + \frac{1}{\bar{B}}\frac{\partial \bar{B}}{\partial \xi}\bar{U} - k_\eta \bar{A}^2\bar{W} + \frac{1}{2}\bar{\lambda}^2\bar{\theta}_2^2\right) = \frac{h^2}{a^2A^2}\bar{\varepsilon}_y;$$

$$\gamma_{xy} = \frac{1}{A}\frac{\partial V}{\partial x} + \frac{1}{B}\frac{\partial U}{\partial y} - \frac{1}{AB}\frac{\partial B}{\partial x}V + \theta_1\theta_2 =$$

$$= \frac{h^2}{a^2A^2}\left(\bar{\lambda}\frac{\partial \bar{V}}{\partial \xi} + \bar{\lambda}\frac{\partial \bar{U}}{\partial \eta} - \bar{\lambda}\frac{1}{\bar{B}}\frac{\partial \bar{B}}{\partial \xi}\bar{V} + \bar{\lambda}\bar{\theta}_1\bar{\theta}_2\right) = \frac{h^2}{a^2A^2}\bar{\lambda}\bar{\gamma}_{xy};$$

$$\gamma_{xz} = kf(z)\left[\Psi_x - \theta_1\right] = \frac{h}{aA}kf(\bar{z})\left[\bar{\Psi}_x - \bar{\theta}_1\right] = \frac{h}{aA}\bar{\gamma}_{xz};$$

$$\gamma_{yz} = kf(z)\left[\Psi_y - \theta_2\right] = \frac{h}{aA}\bar{\lambda}kf(\bar{z})\left[\bar{\Psi}_y - \bar{\theta}_2\right] = \frac{h}{aA}\bar{\gamma}_{yz};$$

$$\chi_1 = \frac{1}{A}\frac{\partial \Psi_x}{\partial x} = \frac{1}{A}\frac{h}{Aa^2}\frac{\partial \bar{\Psi}_x}{\partial \xi} = \frac{h}{A^2a^2}\bar{\chi}_1;$$

$$\chi_2 = \frac{1}{B}\frac{\partial \Psi_y}{\partial y} + \frac{1}{AB}\frac{\partial B}{\partial x}\Psi_x =$$

$$= \frac{h}{A^2a^2}\left(\bar{\lambda}^2\frac{\partial \bar{\Psi}_y}{\partial \eta} + \frac{1}{\bar{B}}\frac{\partial \bar{B}}{\partial \xi}\bar{\Psi}_x\right) = \frac{h}{A^2a^2}\bar{\chi}_2;$$

$$2\chi_{12} = \frac{1}{A}\frac{\partial \Psi_y}{\partial x} + \frac{1}{B}\frac{\partial \Psi_x}{\partial y} - \frac{1}{AB}\frac{\partial B}{\partial x}\Psi_y =$$

$$= \frac{h}{a^2A^2}\left(\bar{\lambda}\frac{\partial \bar{\Psi}_y}{\partial \xi} + \bar{\lambda}\frac{\partial \bar{\Psi}_x}{\partial \eta} - \bar{\lambda}\frac{1}{\bar{B}}\frac{\partial \bar{B}}{\partial \xi}\bar{\Psi}_y\right) = \frac{h}{a^2A^2}\bar{\lambda}2\bar{\chi}_{12}.$$

Under linear elastic deformation, the physical relations for orthotropic rotational shells have the form (3), let us pass in them to the dimensionless parameters (9)

$$\sigma_x = \frac{E_1}{1-\mu_{12}\mu_{21}}\left[\varepsilon_x + \mu_{21}\varepsilon_y + z(\chi_1 + \mu_{21}\chi_2)\right] =$$

$$= \frac{E_1}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\left[\bar{\varepsilon}_x + \mu_{21}\bar{\varepsilon}_y + \bar{z}(\bar{\chi}_1 + \mu_{21}\bar{\chi}_2)\right] =$$

$$= \frac{E_1}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\bar{\sigma}_x;$$

$$\sigma_y = \frac{E_2}{1-\mu_{12}\mu_{21}}\left[\varepsilon_y + \mu_{12}\varepsilon_x + z(\chi_2 + \mu_{12}\chi_1)\right] =$$

$$= \frac{E_2}{1-\mu_{12}\mu_{21}} \frac{h^2}{a^2A^2}\left[\bar{\varepsilon}_y + \mu_{12}\bar{\varepsilon}_x + \bar{z}\left(\bar{\chi}_2 + \mu_{12}\bar{\chi}_1\right)\right] =$$

$$= \frac{E_2}{1-\mu_{12}\mu_{21}} \frac{h^2}{a^2A^2}\bar{\sigma}_y;$$

$$\tau_{xy} = G_{12}\left[\gamma_{xy} + 2z\chi_{12}\right] =$$

$$= G_{12}\frac{h^2}{a^2A^2}\bar{\lambda}\left[\bar{\gamma}_{xy} + 2\bar{z}\bar{\chi}_{12}\right] = G_{12}\frac{h^2}{a^2A^2}\bar{\tau}_{xy};$$

$$\tau_{xz} = G_{13}kf(z)\left[\Psi_x - \theta_1\right] =$$

$$= G_{13}\frac{h^2}{a^2A^2}kf(\bar{z})\bar{A}\left[\bar{\Psi}_x - \bar{\theta}_1\right] = G_{13}\frac{h^2}{a^2A^2}\bar{\tau}_{xz};$$

$$\tau_{yz} = G_{23}kf(z)\left[\Psi_y - \theta_2\right] =$$

$$= G_{23}\frac{h^2}{a^2A^2}kf(\bar{z})\bar{A}\bar{\lambda}\left[\bar{\Psi}_y - \bar{\theta}_2\right] = G_{23}\frac{h^2}{a^2A^2}\bar{\tau}_{yz}.$$

Here

$$f(\bar{z}) = 6\left(\frac{1}{4} - \bar{z}^2\right).$$

The functional of the total potential energy of deformation of the shell is written as follows:

$$E_p = \frac{E_1h^7}{2a^4A^4(1-\mu_{12}\mu_{21})}\int_0^1\int_0^1\Big\{\bar{\varepsilon}_x^2 + \bar{G}_2\bar{\varepsilon}_y^2 +$$

$$+\left(\mu_{21} + \bar{G}_2\mu_{12}\right)\bar{\varepsilon}_x\bar{\varepsilon}_y + \bar{G}_{12}\bar{\lambda}^2\bar{\gamma}_{xy}^2 +$$

$$\frac{1}{12}\left(\bar{\chi}_1^2 + \bar{G}_2\bar{\chi}_2^2 + \left(\mu_{21} + \bar{G}_2\mu_{12}\right)\bar{\chi}_1\bar{\chi}_2 + 4\bar{G}_{12}\bar{\lambda}^2\bar{\chi}_{12}^2\right) +$$

$$+\bar{G}_{13}k\bar{A}^2\left(\bar{\Psi}_x - \bar{\theta}_1\right)^2 + \bar{G}_{23}k\bar{A}^2\bar{\lambda}^2\left(\bar{\Psi}_y - \bar{\theta}_2\right)^2 -$$

$$-2\left(1-\mu_{12}\mu_{21}\right)\bar{P}\bar{W}\Big\}\bar{A}\bar{B}d\xi d\eta = \frac{E_1h^7}{2a^4A^4(1-\mu_{12}\mu_{21})}\bar{E}_p. \quad (10)$$

Thus, only the dimensionless summands will be under the integral sign.

If necessary, it is possible to obtain dimensionless values for the forces and moments:

$$N_x = \frac{E_1h}{1-\mu_{12}\mu_{21}}\left(\varepsilon_x + \mu_{21}\varepsilon_y\right) = \frac{E_1h}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\left(\bar{\varepsilon}_x + \mu_{21}\bar{\varepsilon}_y\right) =$$

$$= \frac{E_1h}{\left(1-\mu_{12}\mu_{21}\right)\bar{A}^2}\left(\bar{\varepsilon}_x + \mu_{21}\bar{\varepsilon}_y\right) = E_1h\bar{N}_x;$$

$$N_y = \frac{E_2h}{1-\mu_{12}\mu_{21}}\left(\varepsilon_y + \mu_{12}\varepsilon_x\right) =$$

$$= \frac{E_2h}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\left(\bar{\varepsilon}_y + \mu_{12}\bar{\varepsilon}_x\right) = E_2h\bar{N}_y;$$

$$N_{xy} = N_{yx} = G_{12}h\gamma_{xy} = G_{12}h\frac{h^2}{a^2A^2}\bar{\lambda}\bar{\gamma}_{xy} =$$

$$= \frac{G_{12}h}{\bar{A}^2}\bar{\lambda}\bar{\gamma}_{xy} = G_{12}h\bar{N}_{xy};$$

$$M_x = \frac{E_1}{1-\mu_{12}\mu_{21}}\left[\left(\frac{h^3}{12}\right)\left(\chi_1 + \mu_{21}\chi_2\right)\right] =$$

$$= \frac{E_1h^3}{12\left(1-\mu_{12}\mu_{21}\right)}\frac{h}{a^2A^2}\left(\bar{\chi}_1 + \mu_{21}\bar{\chi}_2\right) =$$

$$= \frac{E_1h^3}{12\left(1-\mu_{12}\mu_{21}\right)\bar{A}^2}\left(\bar{\chi}_1 + \mu_{21}\bar{\chi}_2\right) = E_1h^2\bar{M}_x;$$

$$M_y = \frac{E_2}{1-\mu_{12}\mu_{21}}\left[\left(\frac{h^3}{12}\right)\left(\chi_2 + \mu_{12}\chi_1\right)\right] =$$

$$= \frac{E_2h^3}{12\left(1-\mu_{12}\mu_{21}\right)}\frac{h}{a^2A^2}\left(\bar{\chi}_2 + \mu_{12}\bar{\chi}_1\right) = E_2h^2\bar{M}_y;$$

$$M_{xy} = M_{yx} = G_{12}\frac{h^3}{6}\chi_{12} = G_{12}\frac{h^2}{6\bar{A}^2}\bar{\lambda}\bar{\chi}_{12} = G_{12}h^2\bar{M}_{xy};$$

$$Q_x = G_{13}kh\left(\Psi_x - \theta_1\right) = G_{13}k\frac{h}{\bar{A}}\left(\bar{\Psi}_x - \bar{\theta}_1\right) = G_{13}h\bar{Q}_x;$$

$$Q_y = G_{23}kh\left(\Psi_y - \theta_2\right) = G_{23}k\frac{h}{\bar{A}}\bar{\lambda}\left(\bar{\Psi}_y - \bar{\theta}_2\right) = G_{23}h\bar{Q}_y.$$

3.3. Dimensionless Parameters for Stiffened Orthotropic Shells

In addition to those mentioned above, it is necessary to introduce the additional dimensionless parameters for reinforced shells

$$\bar{F}_x = \frac{F_x}{h}, \quad \bar{F}_y = \frac{F_y}{h}, \quad \bar{S}_x = \frac{S_x}{h^2}, \quad \bar{S}_y = \frac{S_y}{h^2},$$

$$\bar{J}_x = \frac{J_x}{h^3}, \quad \bar{J}_y = \frac{J_y}{h^3}, \quad \bar{H} = \frac{H}{h},$$

$$f(z) = -\frac{6}{\left(1+\bar{H}\right)^2}\left(\bar{z} + \frac{1}{2}\right)\left(\bar{z} - \frac{1}{2} - \bar{H}\right) = f(\bar{z}).$$

$$\bar{x}_j = \frac{x_j}{a}, \quad \bar{r}_j = \frac{r_j}{a}, \quad \bar{h}_j = \frac{h_j}{h}, \quad \bar{a}_j = \frac{a_j}{a}, \quad \bar{b}_j = \frac{b_j}{a}, \quad (11)$$

$$\bar{y}_i = \frac{y_i}{b}, \quad \bar{r}_i = \frac{r_i}{b}, \quad \bar{h}_i = \frac{h_i}{h}, \quad \bar{c}_i = \frac{c_i}{b}, \quad \bar{d}_i = \frac{d_i}{b}.$$

The thickness of this whole structure will be equal to $1+\bar{H}$.

The functional of the total potential energy of deformation will have the following form

$$E_p = \frac{E_1h^7}{2a^4A^4(1-\mu_{12}\mu_{21})}\int_0^1\int_0^1\Big\{\left(1+\bar{F}_x\right)\bar{\varepsilon}_x^2 + \bar{G}_2\left(1+\bar{F}_y\right)\bar{\varepsilon}_y^2 +$$

$$+\mu_{21}\left(2+\bar F_x+\bar F_y\right)\bar\varepsilon_x\bar\varepsilon_y+\frac{1}{2}\bar G_{12}\left(2+\bar F_x+\bar F_y\right)\bar\lambda^2\bar\gamma_{xy}^2+$$

$$+\bar G_{13}k\bar A^2\left(1+\bar F_x\right)\left(\bar\Psi_x-\bar\theta_1\right)^2+$$

$$+\bar G_{23}k\bar A^2\bar\lambda^2\left(1+\bar F_y\right)\left(\bar\Psi_y-\bar\theta_2\right)^2+$$

$$+2\bar S_x\bar\varepsilon_x\bar\chi_1+\mu_{21}\left(\bar S_x+\bar S_y\right)\bar\varepsilon_x\bar\chi_2+\mu_{21}\left(\bar S_x+\bar S_y\right)\bar\varepsilon_y\bar\chi_1+$$

$$+2\bar G_2\bar S_y\bar\varepsilon_y\bar\chi_2+2\bar G_{12}\left(\bar S_x+\bar S_y\right)\bar\lambda^2\bar\gamma_{xy}\bar\chi_{12}+$$

$$+\left(\frac{1}{12}+\bar J_x\right)\bar\chi_1^2+\bar G_2\left(\frac{1}{12}+\bar J_y\right)\bar\chi_2^2+$$

$$+\mu_{21}\left(\frac{1}{6}+\bar J_x+\bar J_y\right)\bar\chi_1\bar\chi_2+2\bar G_{12}\left(\frac{1}{6}+\bar J_x+\bar J_y\right)\bar\lambda^2\bar\chi_{12}^2-$$

$$\left.-2\left(1-\mu_{12}\mu_{21}\right)\bar P\bar W\right\}\bar A\bar B d\xi d\eta=$$

$$=\frac{E_1h^7}{2a^4A^4\left(1-\mu_{12}\mu_{21}\right)}\bar E_p. \qquad (12)$$

The expressions of forces and moments for reinforced orthotropic shells will be written as follows:

$$N_x=\frac{E_1}{1-\mu_{12}\mu_{21}}\left[\left(h+F_x\right)\left(\varepsilon_x+\mu_{21}\varepsilon_y\right)+S_x\left(\chi_1+\mu_{21}\chi_2\right)\right]=$$

$$=\frac{E_1h}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\times$$

$$\times\left[\left(1+\bar F_x\right)\left(\bar\varepsilon_x+\mu_{21}\bar\varepsilon_y\right)+\bar S_x\left(\bar\chi_1+\mu_{21}\bar\chi_2\right)\right]=E_1h\bar N_x,$$

$$N_y=\frac{E_2}{1-\mu_{12}\mu_{21}}\left[\left(h+F_y\right)\left(\varepsilon_y+\mu_{12}\varepsilon_x\right)+S_y\left(\chi_2+\mu_{12}\chi_1\right)\right]=$$

$$=\frac{E_2h}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\times$$

$$\times\left[\left(1+\bar F_y\right)\left(\bar\varepsilon_y+\mu_{12}\bar\varepsilon_x\right)+\bar S_y\left(\bar\chi_2+\mu_{12}\bar\chi_1\right)\right]=E_2h\bar N_y,$$

$$N_{xy}=G_{12}\left[\left(h+F_y\right)\gamma_{xy}+2S_y\chi_{12}\right]=$$

$$=G_{12}h\bar\lambda\frac{h^2}{a^2A^2}\left[\left(1+\bar F_y\right)\bar\gamma_{xy}+2\bar S_y\bar\chi_{12}\right]=G_{12}h\bar N_{xy},$$

$$N_{yx}=G_{12}\left[\left(h+F_x\right)\gamma_{xy}+2S_x\chi_{12}\right]=$$

$$=G_{12}h\bar\lambda\frac{h^2}{a^2A^2}\left[\left(1+\bar F_x\right)\bar\gamma_{xy}+2\bar S_x\bar\chi_{12}\right]=G_{12}h\bar N_{yx},$$

$$M_x=\frac{E_1}{1-\mu_{12}\mu_{21}}\left[S_x\left(\varepsilon_x+\mu_{21}\varepsilon_y\right)+\left(\frac{h^3}{12}+J_x\right)\left(\chi_1+\mu_{21}\chi_2\right)\right]=$$

$$=\frac{E_1h^2}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\times$$

$$\times\left[\bar S_x\left(\bar\varepsilon_x+\mu_{21}\bar\varepsilon_y\right)+\left(\frac{1}{12}+\bar J_x\right)\left(\bar\chi_1+\mu_{21}\bar\chi_2\right)\right]=E_1h^2\bar M_x,$$

$$M_y=\frac{E_2}{1-\mu_{12}\mu_{21}}\left[S_y\left(\varepsilon_y+\mu_{12}\varepsilon_x\right)+\left(\frac{h^3}{12}+J_y\right)\left(\chi_2+\mu_{12}\chi_1\right)\right]=$$

$$=\frac{E_2h^2}{1-\mu_{12}\mu_{21}}\frac{h^2}{a^2A^2}\times$$

$$\times\left[\bar S_y\left(\bar\varepsilon_y+\mu_{12}\bar\varepsilon_x\right)+\left(\frac{1}{12}+\bar J_y\right)\left(\bar\chi_2+\mu_{12}\bar\chi_1\right)\right]=E_2h^2\bar M_y,$$

$$M_{xy}=G_{12}\left[S_y\gamma_{xy}+2\left(\frac{h^3}{12}+J_y\right)\chi_{12}\right]=$$

$$=G_{12}h^2\bar\lambda\frac{h^2}{a^2A^2}\left[\bar S_y\bar\gamma_{xy}+2\left(\frac{1}{12}+\bar J_y\right)\bar\chi_{12}\right]=G_{12}h^2\bar M_{xy},$$

$$M_{yx}=G_{12}\left[S_x\gamma_{xy}+2\left(\frac{h^3}{12}+J_x\right)\chi_{12}\right]=$$

$$=G_{12}h^2\bar\lambda\frac{h^2}{a^2A^2}\left[\bar S_x\bar\gamma_{xy}+2\left(\frac{1}{12}+\bar J_x\right)\bar\chi_{12}\right]=G_{12}h^2\bar M_{yx},$$

$$Q_x=G_{13}k\left(h+F_x\right)\left(\Psi_x-\theta_1\right)=$$

$$=G_{13}hk\frac{h}{aA}\left(1+\bar F_x\right)\left(\bar\Psi_x-\bar\theta_1\right)=G_{13}h\bar Q_x,$$

$$Q_y=G_{23}k\left(h+F_y\right)\left(\Psi_y-\theta_2\right)=$$

$$=G_{23}hk\frac{h}{aA}\bar\lambda\left(1+\bar F_y\right)\left(\bar\Psi_y-\bar\theta_2\right)=G_{23}h\bar Q_y.$$

3.4. Dimensionless Parameters for Shallow Isotropic Shells when Solving Dynamics and Thermoelasticity Problems

When solving mixed-mode equations, we introduce the stress function in the shell middle surface $\Phi(x,y)$. In dimensionless parameters, it has the form $\bar\Phi(\xi,\eta)=\dfrac{\Phi(x,y)}{Eh^2}$.

When solving dynamics problems, we introduce the dimensionless time parameter [37]

$$\bar t=\frac{h}{a^2}\sqrt{\frac{E}{\left(1-\mu^2\right)\rho}}\cdot t,\quad \rho=\frac{\gamma}{g}.$$

Here γ, g is the specific weight of the shell material and gravitational acceleration.

When solving thermoelasticity problems, we introduce a dimensionless temperature parameter [37] $\bar{T} = \frac{\alpha a^2}{h^2} T$, where α is the material thermal expansion coefficient.

3.5. Some Special Cases

Let us consider a special case for unstiffened shallow isotropic shells of rectangular planform. In this case, the Lamé parameters will be $A = 1, B = 1$ and the dimensionless parameters will take the following form

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \bar{\lambda} = \frac{a}{b},$$
$$k_\xi = \frac{a^2 k_x}{h}, \quad k_\eta = \frac{b^2 k_y}{h},$$
$$\bar{U} = \frac{aU}{h^2}, \quad \bar{V} = \frac{bV}{h^2}, \quad \bar{W} = \frac{W}{h}, \tag{13}$$
$$\bar{\Psi}_x = \frac{\Psi_x a}{h}, \quad \bar{\Psi}_y = \frac{\Psi_y b}{h},$$
$$\bar{\sigma}_x = \frac{\sigma_x a^2}{Eh^2}, \quad \bar{\sigma}_y = \frac{\sigma_y a^2}{Eh^2}, \quad \bar{\tau}_{xy} = \frac{\tau_{xy} a^2}{G_{12} h^2},$$
$$\bar{P} = \frac{a^4 q}{h^4 E}, \quad \bar{a} = \frac{a}{h}, \quad \bar{z} = \frac{z}{h}.$$

Let us note that in this variant of dimensionless parameters, in addition to setting the Lamé parameters and executing simplifications connected with the material isotropy, the expressions for the shell main curvatures [37] are chosen in a different way.

The expressions for strains will take the following form:

$$\varepsilon_x = \frac{\partial U}{\partial x} - k_x W + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 =$$
$$= \frac{h^2}{a^2}\left(\frac{\partial \bar{U}}{\partial \xi} - k_\xi \bar{W} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \xi}\right)^2\right) = \frac{h^2}{a^2}\bar{\varepsilon}_x;$$

$$\varepsilon_y = \frac{\partial V}{\partial y} - k_y W + \frac{1}{2}\left(\frac{\partial W}{\partial y}\right)^2 =$$
$$= \frac{h^2}{a^2}\bar{\lambda}^2\left(\frac{\partial \bar{V}}{\partial \eta} - k_\eta \bar{W} + \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \eta}\right)^2\right) = \frac{h^2}{a^2}\bar{\lambda}^2\bar{\varepsilon}_y;$$

$$\gamma_{xy} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial W}{\partial x}\frac{\partial W}{\partial y} =$$
$$= \frac{h^2}{a^2}\bar{\lambda}\left(\frac{\partial \bar{V}}{\partial \xi} + \frac{\partial \bar{U}}{\partial \eta} + \frac{\partial \bar{W}}{\partial \xi}\frac{\partial \bar{W}}{\partial \eta}\right) = \frac{h^2}{a^2}\bar{\lambda}\bar{\gamma}_{xy};$$

$$\gamma_{xz} = k\ f(z)\left[\Psi_x - \frac{\partial W}{\partial x}\right] =$$
$$= \frac{h}{a}k\ f(\bar{z})\left[\bar{\Psi}_x - \frac{\partial \bar{W}}{\partial \xi}\right] = \frac{h}{a}\bar{\gamma}_{xz};$$

$$\gamma_{yz} = k\ f(z)\left[\Psi_y - \frac{\partial W}{\partial y}\right] =$$
$$= \frac{h}{a}\bar{\lambda}k\ f(\bar{z})\left[\bar{\Psi}_y - \frac{\partial \bar{W}}{\partial \eta}\right] = \frac{h}{a}\bar{\lambda}\bar{\gamma}_{yz};$$

$$\chi_1 = \frac{\partial \Psi_x}{\partial x} = \frac{h}{a^2}\frac{\partial \bar{\Psi}_x}{\partial \xi} = \frac{h}{a^2}\bar{\chi}_1;$$

$$\chi_2 = \frac{\partial \Psi_y}{\partial y} = \frac{h}{a^2}\bar{\lambda}^2\frac{\partial \bar{\Psi}_y}{\partial \eta} = \frac{h}{a^2}\bar{\lambda}^2\bar{\chi}_2;$$

$$2\chi_{12} = \frac{\partial \Psi_y}{\partial x} + \frac{\partial \Psi_x}{\partial y} = \frac{h}{a^2}\bar{\lambda}\left(\frac{\partial \bar{\Psi}_y}{\partial \xi} + \frac{\partial \bar{\Psi}_x}{\partial \eta}\right) = \frac{h}{a^2}\bar{\lambda}2\bar{\chi}_{12}.$$

Under linear elastic deformation, the physical relations for isotropic rotational shells will have the form (3) at

$$E_1 = E_2 = E, \ \mu_{12} = \mu_{21} = \mu, \qquad G_{12} = G_{13} = G_{23} = \frac{E}{2(1+\mu)}.$$

Let us pass in them to the dimensionless parameters (13)

$$\sigma_x = \frac{E}{1-\mu^2}\left[\varepsilon_x + \mu\varepsilon_y + z(\chi_1 + \mu\chi_2)\right] =$$
$$= \frac{E}{1-\mu^2}\frac{h^2}{a^2}\left[\bar{\varepsilon}_x + \mu\bar{\lambda}^2\bar{\varepsilon}_y + \bar{z}\left(\bar{\chi}_1 + \mu\bar{\lambda}^2\bar{\chi}_2\right)\right] = E\frac{h^2}{a^2}\bar{\sigma}_x;$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\varepsilon_y + \mu\varepsilon_x + z(\chi_2 + \mu\chi_1)\right] =$$
$$= \frac{E}{1-\mu^2}\frac{h^2}{a^2}\left[\bar{\lambda}^2\bar{\varepsilon}_y + \mu\bar{\varepsilon}_x + \bar{z}\left(\bar{\lambda}^2\bar{\chi}_2 + \mu\bar{\chi}_1\right)\right] = E\frac{h^2}{a^2}\bar{\sigma}_y;$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\left[\gamma_{xy} + 2z\chi_{12}\right] =$$
$$= \frac{E}{2(1+\mu)}\bar{\lambda}\frac{h^2}{a^2}\left[\bar{\gamma}_{xy} + 2\bar{z}\bar{\chi}_{12}\right] = E\frac{h^2}{a^2}\bar{\tau}_{xy};$$

$$\tau_{xz} = G_{13}\gamma_{xz} = G_{13}\frac{h}{a}\bar{\gamma}_{xz}; \ \tau_{yz} = G_{23}\gamma_{yz} = G_{23}\frac{h}{a}\bar{\lambda}\bar{\gamma}_{yz}.$$

The functional of the total potential energy of deformation of the shallow shell of rectangular planform, with (13) taken into account, will have the form [37]

$$E_p = \frac{Eh}{2(1-\mu^2)}\int_0^a\int_0^b\left\{\varepsilon_x^2 + \varepsilon_y^2 + 2\mu\varepsilon_x\varepsilon_y + \bar{G}\gamma_{xy}^2 + \bar{G}k\left(\Psi_x - \theta_1\right)^2 +\right.$$
$$+ \bar{G}k\left(\Psi_y - \theta_2\right)^2 +$$
$$\left.+\frac{h^2}{12}\left(\chi_1^2 + \chi_2^2 + 2\mu\chi_1\chi_2 + 4\bar{G}\chi_{12}^2\right) - 2\frac{q(1-\mu^2)}{Eh}W\right\}dxdy =$$

$$=\frac{Eh^5}{2\left(1-\mu^2\right)a^4}\int_0^1\int_0^1\left\{\bar{\varepsilon}_x^2+\bar{\lambda}^4\bar{\varepsilon}_y^2+2\mu\bar{\lambda}^2\bar{\varepsilon}_x\bar{\varepsilon}_y+\bar{G}\bar{\lambda}^2\bar{\gamma}_{xy}^2+\right.$$

$$+\bar{G}k\left(\bar{\Psi}_x-\bar{\theta}_1\right)^2+\bar{G}k\bar{\lambda}^2\left(\bar{\Psi}_y-\bar{\theta}_2\right)^2+$$

$$\left.+\frac{h^2}{12}\left(\bar{\chi}_1^2+\bar{\lambda}^4\bar{\chi}_2^2+2\mu\bar{\lambda}^2\bar{\chi}_1\bar{\chi}_2+4\bar{G}\bar{\lambda}^2\bar{\chi}_{12}^2\right)-2\bar{P}\left(1-\mu^2\right)\bar{W}\right\}\times$$

$$\times\,d\xi d\eta=\frac{Eh^5}{2\left(1-\mu^2\right)a^4}\bar{E}_p.$$

Here

$$\bar{G}=\frac{G_{12}\left(1-\mu^2\right)}{E}.$$

3.6. Further Approach to Solving Problems of Strength and Stability of Isotropic and Orthotropic Shell Structures in Dimensionless and Dimensional Parameters

In order to minimize the functional of the total potential energy of deformation, we use the Ritz method. In order to solve problems in dimensional parameters, let us imagine the sought-for functions as follows:

$$U=U\left(x,y\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}U_{(p-1)\sqrt{N}+l}\cdot X1\left(p\right)Y1\left(l\right),$$

$$V=V\left(x,y\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}V_{(p-1)\sqrt{N}+l}\cdot X2\left(p\right)Y2\left(l\right),$$

$$W=W\left(x,y\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}W_{(p-1)\sqrt{N}+l}\cdot X3\left(p\right)Y3\left(l\right),\quad(14)$$

$$\Psi_x=\Psi_x\left(x,y\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}PS_{(p-1)\sqrt{N}+l}\cdot X4\left(p\right)Y4\left(l\right),$$

$$\Psi_y=\Psi_y\left(x,y\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}PN_{(p-1)\sqrt{N}+l}\cdot X5\left(p\right)Y5\left(l\right),$$

where $U_i, V_i, W_i, PS_i, PN_i$ are unknown numerical coefficients and $X1(I)-X5(I),\ Y1(I)-Y5(I)$ are the known approximating functions of the *x* and *y* arguments, that are satisfactory to the set boundary conditions on the shell contour, *N* is the quantity of expansion terms.

In order to solve problems in dimensionless parameters, the sought-for functions will be

$$\bar{U}=\bar{U}\left(\xi,\eta\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\bar{U}_{(p-1)\sqrt{N}+l}\cdot\bar{X}1\left(p\right)\bar{Y}1\left(l\right),$$

$$\bar{V}=\bar{V}\left(\xi,\eta\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\bar{V}_{(p-1)\sqrt{N}+l}\cdot\bar{X}2\left(p\right)\bar{Y}2\left(l\right),$$

$$\bar{W}=\bar{W}\left(\xi,\eta\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\bar{W}_{(p-1)\sqrt{N}+l}\cdot\bar{X}3\left(p\right)\bar{Y}3\left(l\right),\quad(15)$$

$$\bar{\Psi}_x=\bar{\Psi}_x\left(\xi,\eta\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\overline{PS}_{(p-1)\sqrt{N}+l}\cdot\bar{X}4\left(p\right)\bar{Y}4\left(l\right),$$

$$\bar{\Psi}_y=\bar{\Psi}_y\left(\xi,\eta\right)=\sum_{p=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\overline{PN}_{(p-1)\sqrt{N}+l}\cdot\bar{X}5\left(p\right)\bar{Y}5\left(l\right),$$

where $U_i, V_i, W_i, PS_i, PN_i$ are the unknown dimensionless numerical coefficients, $X1(I)-X5(I),\ Y1(I)-Y5(I)$ are the known approximating functions of the ξ and η arguments, that are satisfactory to the set boundary conditions on the shell contour.

After substituting the expansions of the sought-for functions into the functional (12) and conducting the procedure of the Ritz method, we will obtain a system of nonlinear algebraic equations. The obtained system will be solved with the use of the best parameter continuation method [36, 38]. The accurate application of the best parameter continuation method can be reduced to analyzing an angle between the vectors of increments of unknown parameters from the past and current steps of solution (whose sum of squares at every step gives a square of the increment arc length in the multidimensional space). For example, if an angle between the solution vectors at the previous and current steps turns out to be too wide, then we decrease the arc length parameter by 10 times and roll back the iteration process by two steps (such angle limitation is chosen as 0.0177 rad for the problems presented in this work). If an angle value is within the range of 0.0094 rad to 0.0177 rad, then the arc length remains the same as at the previous step. If an angle value does not exceed 0.0094 rad, then the arc length increases by 1.5 times. Thus, it is possible to achieve the sufficient solution accuracy near the critical points and quickly pass the curve flat portions at the same time.

The accuracy of the Ritz method depends on a number of expansions of the sought-for functions into series. A computational experiment shows that for the shells under consideration, it is sufficient to take *N* = 16 in order to ensure quite high accuracy [39].

Such approach allows us to study strength and stability of shells, bypass critical points of the "load-deflection" curve, obtain values of upper and lower critical loads, find bifurcation points and study the structure postbuckling behavior [36, 38].

4. Calculations

4.1. Characteristics of the Structures under Consideration

All the structures considered below have a hinged-fixed support of the contour and are under the effect of uniformly distributed external static transverse load directed along the normal to the middle surface.

We will further consider the structures made of the following isotropic and orthotropic materials (Table 2).

By using dimensionless parameters, we will study the strength and stability of a whole range of such shells by one calculation. In the Ritz method, we will take *N* = 16.

Table 2

Mechanical characteristics of materials of the structures under consideration

Characteristic	Orthotropic		Isotropic
	Carbon fiber reinforced plastic	Fiberglass	Steel
	M60J/Epoxy	*T-10/UPE22-27*	
E_1, MPa	$3.3 \cdot 10^5$	$0.294 \cdot 10^5$	$2.1 \cdot 10^5$
μ_{12}	0.32	0.123	0.3
E_2, MPa	$0.59 \cdot 10^4$	$1.78 \cdot 10^4$	$2.1 \cdot 10^5$
G_{12}, MPa	$0.39 \cdot 10^4$	$0.301 \cdot 10^4$	$0.807 \cdot 10^5$
G_{13}, MPa	$0.39 \cdot 10^4$	$0.301 \cdot 10^4$	$0.807 \cdot 10^5$
G_{23}, MPa	$0.39 \cdot 10^4$	$0.301 \cdot 10^4$	$0.807 \cdot 10^5$
F_1^+, MPa	1760	508	–
F_1^-, MPa	–780	–209	–
F_2^+, MPa	30	246	–
F_2^-, MPa	–168	–117	–
F_{12}, MPa	39	43	–
σ_T, MPa	–	–	1720
Density, kg/m^3	1500	1800	7800

4.2. Doubly-curved Shallow Shells with Square Planform

Let us first consider such shallow shells of square planform with the following dimensional and dimensionless parameters (Table 3).

The conducted calculations in dimensional parameters for variant 4 of carbon fiber reinforced plastic M60J show, that a critical load of the first loss in stability amounts to $\bar{P}_{cr} = 5198.543$ (the dimensional value $q_{cr} = 0.01323$ MPa), the load of the loss in strength is $\bar{P}_{pr} = 12670.4$ (the dimensional value $q_{pr} = 0.0322$ MPa). These loads will be equal for all the considered variants of shallow shells.

Table 3

Example of dimensional and dimensionless parameters for such shallow shells with square planform

Variant	a, m	R_1, m	h, m	$\bar{a}$	$\bar{R}_1$	k_ξ
1	18	45.3	0.03	600	1510	238.41
2	27	67.95	0.045			
3	36	90.6	0.06			
4	54	135.9	0.09			

A maximum deflection at the loss in stability $\bar{W}_{\max}$ is equal to 0.858, which will be 0.077 m, if it is converted into dimensional parameters. The values obtained by conversion into dimensional parameters for the remaining variants of structures are presented in Table 4.

4.3. Cylindrical Shell Panels

Let us further consider the cylindrical shell panels made of fiberglass. We will make a calculation for a structure with parameters $\bar{a} = 2000, \bar{R} = 540,$ an angle of turn π.

A critical load of the first loss in stability amounts to $\bar{P}_{cr} = 3.554132 \cdot 10^7$ (the dimensional value $q_{cr} = 0.065307$ MPa), a load of the loss in strength is $\bar{P}_{pr} = 2.194 \cdot 10^7$ (the dimensional value $q_{pr} = 0.0403$ MPa). These loads will be equal for all the considered variants of shells. A maximum deflection at the loss in stability $\bar{W}_{\max}$ is equal to 20.62, which will be 0.2062 m, if it is converted into dimensional parameters.

We also made calculations for the same shell in dimensional parameters $(a = 20\text{ m}, R = 5.4\text{ m}, h = 0.01\text{ m})$. The values obtained from the calculation in both dimensionless and dimensional parameters are shown in Table 5.

Table 4

Dimensional and dimensionless calculation results for similar shallow shells with square planform

Parameter (dimensionless)	Value	Parameter (dimensional)	When converted into dimensional variants				Comparison (calculation in dimensional)	
			Var.1	Var.2	Var.3	Var.4	value	Difference, %
Input parameters of the structure								
$\bar{a}$	600	a, m	18	27	36	54	54	–
$\bar{R}_1$	1510	R_1, m	45.3	67.95	90.6	135.9	135.9	–
–	–	h, m	0.03	0.045	0.06	0.09	0.09	–
Calculation results								
$\bar{P}_{cr}$	5198.543	q_{cr}, MPa	0.01323				0.01318	0.4
$\bar{P}_{pr}$	12670	q_{pr}, MPa	0.0322				0.0308	4.3
$\bar{W}_{\max}$	0.858	$W_{\max}$, m	0.026	0.039	0.051	0.077	0.088	14.6

Table 5

Dimensional and dimensionless calculation results for similar cylindrical panels

Parameter (dimension less)	Dimensionless value	Parameter (dimensional)	When converted into dimensional variants				Comparison (calculation in dimensional)	
			Var.1	Var.2	Var.3	Var.4	value	difference, %
		Input parameters of the structure						
$\bar{a}$	2000	a, m	5	10	15	20	20	–
–	–	h, m	0.0025	0.005	0.0075	0.01	0.01	–
		Calculation results						
$\bar{P}_{cr}$	$3.554\cdot 10^7$	q_{cr}, MPa	0.065307				0.066250	1
$\bar{P}_{pr}$	$2.194\cdot 10^7$	q_{pr}, MPa	0.0403				0.0407	1
$\bar{W}_{\max}$	20.62	$W_{\max}$, m	0.0515	0.1031	0.1546	0.2062	0.2192	6

4.4. Truncated Conical Shell Panels

Let us consider truncated unstiffened conical shell panels made of isotropic material (steel). Dimensional and dimensionless parameters are given in Table 6. In order to make it possible to consider shells as such, we took an identical angle of conicity $\theta = 0.78$ rad and an identical angle of turn $b = \pi$ rad. We also observe the relation equality rule $\frac{a}{h}, \frac{a_1}{h}$ for all the variants under consideration. Strength and stability calculation results for these structures are given in Table 7. Here a loss in strength takes place simultaneously with a loss in stability, that is why the values of corresponding loads are united. The comparison with the calculation in dimensional parameters was made with respect to the structure of variant 2.

Table 6

Example of dimensional and dimensionless parameters for similar truncated conical shell panels

Var.	a, m	a_1, m	h, m	$a-a_1$, m	$\bar{a}=a/h$	$\bar{a}_1 = a_1/h$
1	12.5	2.5	0.005	10	2500	500
2	25	5	0.010	20		
3	37.5	7.5	0.015	30		
4	50	10	0.020	40		

5. Analysis

For the shells made of the same material, with identical values of the dimensionless parameters $\bar{A}, \bar{B}, k_\xi, k_\eta$ and the different dimensional values a, b, h, R_1, R_2 (similar shells), the values of the critical load of the loss in stability (or strength) $\bar{P}$ and of the stresses $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$ will be identical. Therefore, the dimensional values of both load and stresses $q, \sigma_x, \sigma_y, \tau_{xy}$ will be identical.

In order to find the dimensional values of deflection, forces and moments, it is necessary to carry out the following actions

$$W = h\bar{W}, \quad N_x = E_1 h\bar{N}_x, \quad N_y = E_2 h\bar{N}_y,$$

$$N_{xy} = G_{12} h\bar{N}_{xy}, \quad N_{yx} = G_{12} h\bar{N}_{yx},$$

$$M_x = E_1 h^2 \bar{M}_x, \quad M_y = E_2 h^2 \bar{M}_y,$$

$$M_{xy} = G_{12} h^2 \bar{M}_{xy}, \quad M_{yx} = G_{12} h^2 \bar{M}_{yx},$$

$$Q_x = G_{13} h\bar{Q}_x, \quad Q_y = G_{23} h\bar{Q}_y.$$

Table 7

Dimensional and dimensionless calculation results for similar truncated conical shell panels

Dimensionless parameter	Dimensionless value	Dimensional parameter	When converted into dimensional variants				Comparison (calculation in dimensional)	
			Var.1	Var.2	Var.3	Var.4	value	difference, %
Input parameters of the structure								
$\bar{a}$	2500	a, m	12.5	25	37.5	50	25	–
–	–	h, m	0.005	0.010	0.015	0.020	0.010	–
Calculation results								
$\bar{P}_{cr} = \bar{P}_{pr}$	$5.07\cdot 10^7$	$q_{cr} = q_p$, MPa	0.2730				0.2847	4
$\bar{W}_{\max 1}$	34.27	$W_{\max 1}$, m	0.1714	0.3427	0.5141	0.6854	0.3515	2

6. Conclusions

The introduction of dimensionless parameters in the course of calculation of shell structures allows to obtain more comprehensive information on the stress and strain state of shells and detect strain features for a whole range of similar shells.

As the conducted studies show, the solutions in dimensional and dimensionless parameters are slightly different from each other. The values of critical loads are relatively equal, but the "load-deflection" curves do not coincide.

In case of a solution in dimensional parameters, the sought-for functions have a different order, for example, the deflections W have the order 10^{-1} and U, V, Ψ_x, Ψ_y have the order 10^{-2}–10^{-3}. In addition, an instrumental error accumulates, if an action is carried out with small numbers. With $N=16,$ the order of a system of algebraic equation is equal to 81.

When solving a problem in dimensionless parameters, the order of sought-for functions becomes more equable and the accumulation of an instrumental error is smaller. However, the order of the load parameter $\bar{P}$ can fundamentally differ from the order of the remaining parameters and can be different for each concrete problem. A correct choice of accuracy parameters for calculation in the best parameter continuation method allows to quite accurately find the critical load values regardless of a difference in orders. Thus, the calculations in dimensionless parameters should be recognized as more reasonable.

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