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# THE SUFFICIENT CONDITION FOR THE POINTWISE CONTACT IN THE TWO-LEAF CURVED ELASTIC ELEMENT OF THE FOOT PROSTHESIS UNDER BENDING

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**Abstract:** The problem of the joint weak bending of the curved leaves of two-leaf spring is considered. This spring is a part of an elastic element of the foot prosthesis. The uniqueness of the solution of the problem is proved. New sufficient condition, which is not stronger than one established before, is found for the pointwise contact between the leaves (when the leaves profiles contact only at one point except for the clamped point). The results obtained contribute to the still incomplete theory of bending of the leaf-springs, which are used in the foot prostheses.

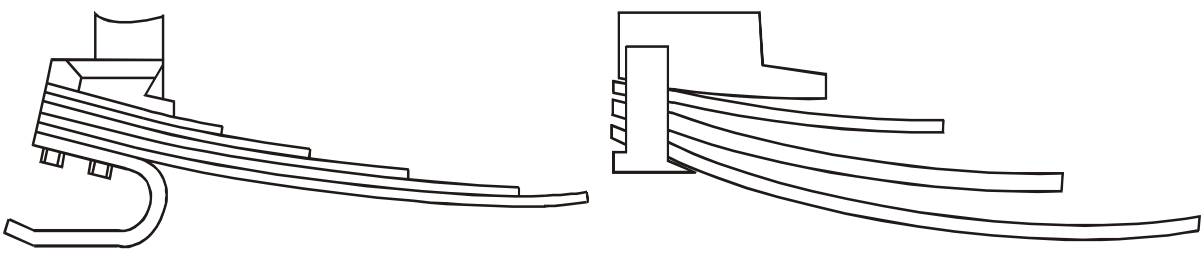
**Key words:** foot prosthesis, elastic element, leaf spring, curved leaves, weak bending, pointwise contact

## Introduction

Some prosthetic foot designs use the elastic element representing the multiple-leaf spring – see Fig. 1a (borrowed from [1]) and Fig. 1b (borrowed from [2]).

Making the design for these prostheses includes the calculation of the spring leaves bending under the given loading (the ground reaction forces).

At first consider one-leaf spring. The following mathematical model of this spring bending has been suggested in [3]. The slender curved beam (leaf) has one edge clamped and the other free (Fig. 2). The beam has constant rectangular cross-section. The natural shape of the beam is described by the function $\varphi(x)$, where $x$ is the length of the beam segment placed between the clamped point and some arbitrary point; $\varphi$ is the angle formed by the tangent to the beam profile (it is assumed that the tangent exists) at these points (Fig. 2); $0 \le x \le \lambda$; $\lambda$ is the beam length. It is assumed below that the function $\varphi(x)$ is continuous, non-decreasing and $\varphi(x) \le \pi/2$ for $0 \le x \le \lambda$. Since $\varphi(0) = 0$, $\varphi(x) \ge 0$. The normal loading with the given density $q(x)$ is applied to the lower side of the beam (Fig. 3; the loading is uniformly



a b

Fig. 1. The elastic elements of the foot prostheses.

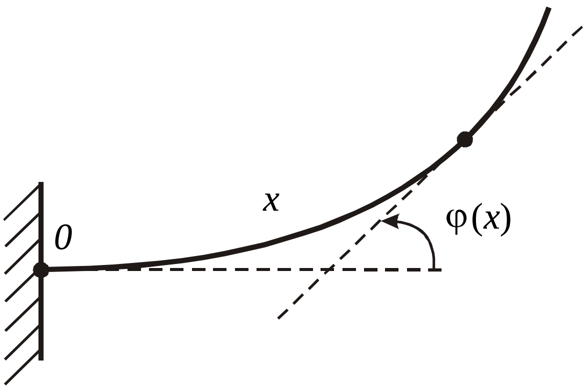



Fig. 2. The model of one-leaf spring; the definition of the function $\varphi(x)$.

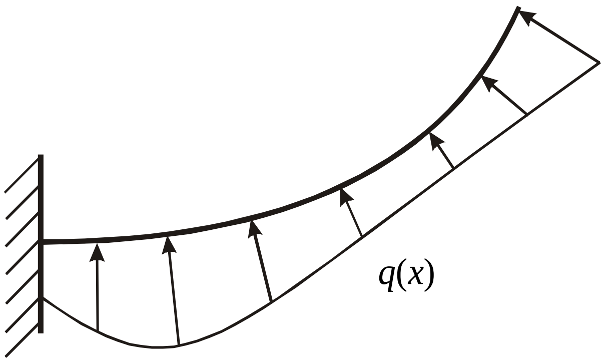



Fig. 3. The loading is normal to the beam.

distributed over the beam width). We assume that $q(x)$ can be represented in the form of

$$u(x)+\sum_{\alpha} U_{\alpha}\delta(x-x_{\alpha}), \quad (1)$$

where $u(x)\geq 0$ is the piecewise continuous function, which is continuous on the right at $x=0$ and on the left at $x>0$; the running integer index $\alpha$ has the finite range; $x_{\alpha}>0$; $U_{\alpha}\geq 0$. It is assumed that the bending of the beam under loading is weak (linear approximation with respect to the loading).

The shape of the beam under bending is described by the normal displacements $y(x)$. Let *A* be the point of the beam without bending with the curvilinear coordinate *x*; let *B* be the position of the same point of the beam under bending. Then $y(x)$ is the projection of the vector $\overline{AB}$ onto the normal to the beam at the point *A* (Fig. 4). The function $y(x)$ (which is to be found) is expressed in terms of $q(x)$ as follows [3]

$$y(x)=k\int_0^{\lambda} G(x,s)q(s)ds, \quad (2)$$

where $k>0$ is the bending compliance of the beam,

$$G(x,s)=G_{*}\big(\max(x,s),\, \min(x,s)\big), \quad (3)$$

$$G_{*}(M,m)=\int_0^{m} g(s,M)g(s,m)ds, \quad (4)$$

$$g(s,\mu)=\int_s^{\mu}\cos(\varphi(\mu)-\varphi(t))\,dt. \quad (5)$$

Note that the function $g(s,\mu)$ is defined and continuous for $0\leq s,\mu\leq\lambda$;

$$g(s,\mu)\begin{cases}\geq 0 & (\mu\geq s),\\ \leq 0 & (\mu\leq s),\\ =0 & (\mu=s);\end{cases} \quad (6)$$

$$G(x,s)\geq 0; \quad (7)$$

$$G(\mu,\mu)>0 \ \text{ for } \ \mu>0. \quad (8)$$

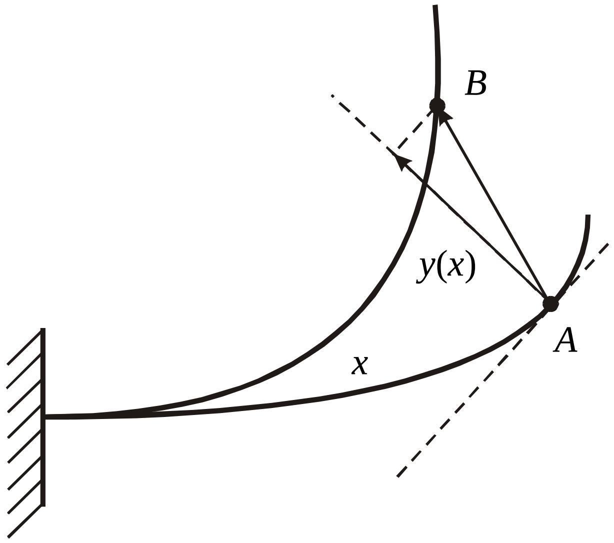



Fig. 4. The definition of the function $y(x)$.

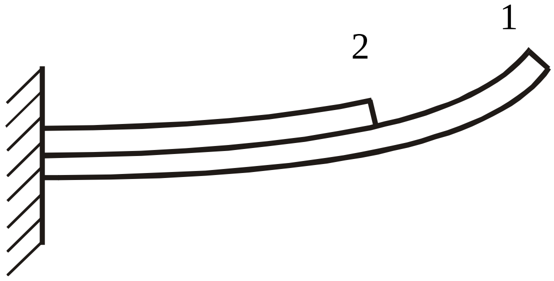



Fig. 5. The model of two-leaf spring.

Integral of type $\int_b^c h(s)\delta(s-s_*)ds$ is considered to be equal to $h(s_*)$ in the cases $s_* = b$ or $s_* = c$.

Now consider two-leaf spring (this article does not deal with the springs which have more than two leaves). The corresponding model is shown in Fig. 5 (the beams thicknesses are exaggerated for distinctness). Two beams (leaves) are pressed close up to each other (without loading). There is no friction between the beams. The lengths of the beams 1 and 2 are denoted as $L$ and $\ell$ respectively. Function $\varphi(x)$ should meet the above-mentioned requirements for $0 \le x \le L$; function $g(s,\mu)$ is then defined and continuous for $0 \le s, \mu \le L$. The loading with the given density $q(x)$ is applied to the lower side of the beam 1. The beams undergo the weak joint bending (with the unbonded contact).

The shapes of the beams 1 and 2 are described by the functions $y_1(x)$ $(0 \le x \le L)$, $y_2(x)$ $(0 \le x \le \ell)$. It is required to find $y_1(x),\ y_2(x)$. In order to solve this problem, it is convenient to reformulate it so that to regard the density $f(x)$ of the forces of interaction between the plates as the function to be found. The functions $y_1(x),\ y_2(x)$ are expressed in terms of $f(x)$ as follows (see (2)):

$$y_1(x) = k_1 \int_0^L G(x,s)q(s)ds - k_1 \int_0^\ell G(x,s) f(s)ds\,, \tag{9}$$

$$y_2(x) = k_2 \int_0^\ell G(x,s) f(s)ds\,, \tag{10}$$

where $k_1, k_2 > 0$ are the bending compliances of the beams. We assume that $f(x)$ is of type (1). We introduce the notation $r(x) = y_2(x) - y_1(x)$; then using (9), (10) we find

$$r(x) = (k_1 + k_2)\int_0^\ell G(x,s) f(s)ds - k_1 \int_0^L G(x,s)q(s)ds\ . \tag{11}$$

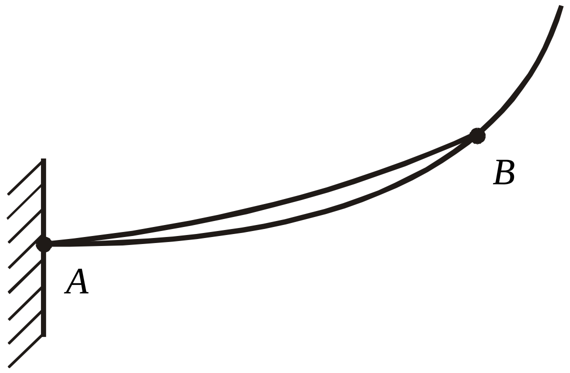



Fig. 6. *A* and *B* are the contact points of the beams profiles.

The beams are impenetrable to each other; this constraint can be formulated as $r(x) \geq 0$ for $0 \leq x \leq \ell$; besides, if $f(x) > 0$ then $r(x) = 0$. Finally we come to the following problem.

*Problem 1.* It is required to find $f(x)$ ($0 \leq x \leq \ell$) which is of type (1) and should satisfy the conditions

$$r(x) \begin{cases} = 0 & (f(x) > 0), \\ \geq 0 & (f(x) = 0), \end{cases} \tag{12}$$

where $r(x)$ is expressed by (11).

The special case $\varphi(x) \equiv 0$ of the problem 1 has been considered in [1]. The solution of this problem has been constructed explicitly and its uniqueness has been proved. It has been found that the inequality

$$\ell^2 \int_{\ell}^{L} (s - \ell) q(s) ds - \int_{0}^{\ell} s(\ell - s)(2\ell - s) q(s) ds \geq 0 \tag{13}$$

is the necessary and sufficient condition for the *pointwise contact* between the beams (Fig. 6), i.e. for the solution of the problem 1 to be

$$f(x) = F\delta(x - \ell). \tag{14}$$

The formulation of the condition for the pointwise contact is of great importance because the multiple-leaf elastic elements of the foot prostheses are calculated on the assumption that the contact is pointwise [4], [5].

The sufficient condition for the pointwise contact in the case of an arbitrary $\varphi(x)$ (which meets the above-mentioned requirements) has been obtained in [3]:

$$q(x) \equiv 0 \text{ for } 0 \leq x < \ell. \tag{15}$$

The uniqueness of the corresponding solution (14) of the problem 1 has not been proved in [3]. Besides, the condition (15) is not necessary for the pointwise contact. In fact, if we consider $q(x) = F_0\delta(x - L) + 2(L/\ell - 1)F_0\delta(x - \ell/2)$, where $F_0 > 0$, then (13) holds (hence, the pointwise contact takes place) but (15) does not hold.

The uniqueness of the solution of the problem 1 is proved (under one additional requirement on $\varphi(x)$) in the present study and new sufficient condition for the pointwise contact is obtained.

**The proof of the uniqueness of the solution of the problem 1**

*Theorem 1.* Suppose that function $\varphi(x)$ meets the above-mentioned requirements and besides, has the piecewise continuous first derivative for $0 \le x \le \ell$. Then the problem 1 may have only one solution.

*Proof.* Let $f(x)$, $f_*(x)$ be the solutions of the problem 1; $0 \le x \le \ell$. We introduce the notation $\rho(x) = f(x) - f_*(x)$. Since $f(x)$, $f_*(x)$ are of type (1), $\rho(x)$ is also of type (1) but $u(x)$, $U_\alpha$ may be negative. We introduce the notation

$$I = \int_0^\ell \rho(x)[\int_0^\ell G(x,s)\rho(s)ds]dx .$$

Then it follows from (11), (12) that $I \le 0$ (either one of the co-factors in the integral over $0 \le x \le \ell$ is equal to zero or these co-factors have different signs). On the other hand, using (3), (4), we obtain

$$I = \int_0^\ell J^2(x)dx , \tag{16}$$

where

$$J(x) = \int_x^\ell g(x,s)\rho(s)ds . \tag{17}$$

It follows from (16) that $I \ge 0$. Hence, $I = 0$. Then, if the above-mentioned properties of $\rho(x)$ and equality (6) are taken into account, it can be proved that $J(x)$ is the continuous function (the simple proofs using the standard methods of mathematical analysis are not adduced in the present study). Then it follows from (16) and the equality $I = 0$ that $J(x) = 0$. Using (5), (17), we obtain

$$J(x) = \int_x^\ell H(s)ds , \tag{18}$$

where

$$H(x) = \int_x^\ell \cos\big(\varphi(s) - \varphi(x)\big)\rho(s)ds .$$

Taking the above-mentioned properties of $\rho(x)$ into account, one can prove that $H(x)$ is the piecewise continuous function, which is continuous on the left for $0 < x \le \ell$ and on the right at $x = 0$. Then it follows from (18) and the equality $J(x) = 0$ that $H(x) = 0$, i.e. (the variable is denoted as *t*)

$$\int_t^\ell \cos\big(\varphi(s) - \varphi(t)\big)\rho(s)ds = 0 \tag{19}$$

for $0 \le t \le \ell$. Multiplying (19) by $\varphi'(t)$ and integrating over $x \le t \le \ell$ yields

$$\int_x^\ell \sin\big(\varphi(s) - \varphi(x)\big)\rho(s)ds = 0 . \tag{20}$$

It follows from (19), (20) that

$$\int_x^\ell \cos\varphi(s)\rho(s)ds = \int_x^\ell \sin\varphi(s)\rho(s)ds = 0 .$$

Using these equalities and taking the above-mentioned properties of $\rho(x)$ into account, one can prove that $\rho(x)=0$. Hence, $f(x)=f_*(x)$. This proves the theorem 1.

## The sufficient condition for the pointwise contact

It is not assumed that the function $\varphi(x)$ has the first derivative in the formulation and proofs of the following lemmas 1, 2 and theorem 2.

We introduce the notation

$$\zeta(M,m)=G_*(M,m)-G_*(m,M). \tag{21}$$

This function is defined for $0\le m, M\le L$. It follows from (4) that

$$\zeta(M,m)=-\int_m^M g(s,M)g(s,m)ds. \tag{22}$$

*Lemma 1.* If $M\ge m$ then $\zeta(M,m)\ge 0$; if $M>0$ then $\zeta(M,0)>0$.

*Proof.* Substituting (5) into (22) yields

$$\zeta(M,m)=\iiint_B \cos\big(\varphi(M)-\varphi(t)\big)\cos\big(\varphi(m)-\varphi(\tau)\big)dt\,d\tau\,ds, \tag{23}$$

where $B:(m\le s\le M,\ s\le t\le M,\ m\le\tau\le s)$. The volume of the region *B* is equal to $(M-m)^3$. The statements of lemma 1 then follow from the fact that integrand function in (23) is continuous, non-negative and is not identical with zero.

*Lemma 2.* If $0\le\alpha\le\beta\le\gamma\le\delta\le L$ then $g(\beta,\delta)g(\alpha,\gamma)-g(\beta,\gamma)g(\alpha,\delta)\ge 0$ and $g(\delta,\beta)g(\gamma,\alpha)-g(\gamma,\beta)g(\delta,\alpha)\ge 0$.

The first inequality of lemma 2 has been proved in [3]. The proof of the second inequality can be performed analogously.

We introduce the notations

$$P=\frac{1}{G(\ell,\ell)}\int_0^L G(\ell,x)q(x)dx, \tag{24}$$

(it follows from (8) that denominator is not equal to zero),

$$c_1=Pg(0,\ell)-\int_0^L g(0,x)q(x)dx, \tag{25}$$

$$c_2=P\sin\varphi(\ell)-\int_0^L \sin\varphi(x)q(x)dx. \tag{26}$$

*Theorem* 2. If

$$c_1\ge 0,\ c_2\ge 0 \tag{27}$$

then the function

$$f(x)=\frac{k_1}{k_1+k_2}P\delta(x-\ell) \tag{28}$$

is the solution of the problem 1.

*Proof.* Using (7) and the fact that $q(x)$ is of type (1), we obtain from (24) that $P\ge 0$. Hence, $f(x)$ is of type (1). Then (12) should be proved. Substituting (28) into (11) yields

$$r(x)/k_1=PG(x,\ell)-\int_0^L G(x,s)q(s)ds. \tag{29}$$

The inequality $f(x) > 0$ may hold, according to (28), only at $x = \ell$. It follows from (24), (29) that $r(\ell) = 0$. Hence, we should establish that $r(x) \geq 0$ for $0 \leq x < \ell$ in order for (12), and consequently the theorem 2, to be proved.

Using the functions $G_*(M, m)$ and $\zeta(M, m)$ (see (3) and (21)), we rewrite (29) in the form

$$r(x)/k_1 = PG_*(\ell, x) - \int_0^x \zeta(x,s)q(s)ds - \int_0^L G_*(s,x)q(s)ds .$$

Using this representation of $r(x)$, we compose the expression $\zeta(\ell,0)r(x) - \zeta(x,0)r(\ell)$; it is equal to $\zeta(\ell,0)r(x)$ because $r(\ell) = 0$. After the transformations we obtain

$$\zeta(\ell,0)r(x)/k_1 = I_1(x) + I_2(x) + I_3(x) , \tag{30}$$

where

$$I_1(x) = \int_{x+0}^{\ell} \zeta(x,0)\zeta(\ell,s)q(s)ds , \tag{31}$$

$$I_2(x) = \int_0^x A(x,s)q(s)ds , \tag{32}$$

$$A(x,s) = \zeta(\ell,s)\zeta(x,0) - \zeta(x,s)\zeta(\ell,0) , \tag{33}$$

$$I_3(x) = P\big(G_*(\ell,x)\zeta(\ell,0) - G_*(\ell,\ell)\zeta(x,0)\big) + $$
$$+ \int_0^L \big(G_*(s,\ell)\zeta(x,0) - G_*(s,x)\zeta(\ell,0)\big)q(s)ds . \tag{34}$$

We will prove that $I_1(x), I_2(x), I_3(x) \geq 0$ for $0 \leq x \leq \ell$.

The inequality

$$I_1(x) \geq 0 \quad \text{for} \quad 0 \leq x \leq \ell \tag{35}$$

follows from (31) and lemma 1.

Using (22), we obtain from (33)

$$A(x,s) = \iint_{B_1} A_1(x,s,t,\tau)dt\, d\tau - \iint_{B_2} A_2(x,s,t,\tau)dt\, d\tau , \tag{36}$$

where

$$0 \leq s \leq x \leq \ell ,\ B_1 : (s \leq t \leq \ell, 0 \leq \tau \leq x) ,\ B_2 : (s \leq t \leq x, 0 \leq \tau \leq \ell) ,$$

$$A_1(x,s,t,\tau) = g(t,s)g(t,\ell)g(\tau,0)g(\tau,x) ,\ A_2(x,s,t,\tau) = g(t,s)g(t,x)g(\tau,0)g(\tau,\ell) .$$

Then we put (36) in the form

$$A(x,s) = \iint_{B_3} A_1(t,\tau)dt\, d\tau + \iint_{B_4} \big(A_1(t,\tau) - A_2(t,\tau)\big)dt\, d\tau + $$
$$+ \frac{1}{2}\iint_{B_5} \big(A_1(t,\tau) - A_2(t,\tau) + A_1(\tau,t) - A_2(\tau,t)\big)dt\, d\tau + \iint_{B_6} \big(A_1(t,\tau) - A_2(\tau,t)\big)dt\, d\tau , \tag{37}$$

where

$$B_3 : (x \leq t \leq \ell, 0 \leq \tau \leq s) ,\ B_4 : (s \leq t \leq x, 0 \leq \tau \leq s) ,$$
$$B_5 : (s \leq t \leq x, s \leq \tau \leq x) ,\ B_6 : (x \leq t \leq \ell, s \leq \tau \leq x) .$$

The variables $x, s$ of the functions $A_1, A_2$ are not written out. Using (6) and lemma 2, one can prove that the integrand functions in all integrals (37) are non-negative in the corresponding regions. Hence, $A(x,s) \geq 0$ for $0 \leq s \leq x \leq \ell$ and it follows from (32) that

$$I_2(x) \geq 0 \quad \text{for} \quad 0 \leq x \leq \ell . \qquad (38)$$

We substitute (4), (22) and (5) into (34) and take (25), (26) into account. After the transformations we find

$$I_3(x) = c_1 C_1(x) + c_2 C_2(x) , \qquad (39)$$

where

$$C_1(x) = \iint_D E_1(x,t,\tau)dt\,d\tau , \; C_2(x) = \iint_D E_2(x,t,\tau)dt\,d\tau , \; D : (0 \leq t \leq \ell, 0 \leq \tau \leq x) ,$$

$$E_1(x,t,\tau) = g(\tau,x)g(t,\ell)\int_\tau^t \cos\varphi(s)ds , \; E_2(x,t,\tau) = g(\tau,x)g(t,\ell)\int_0^t \left( \int_0^\tau \sin(\varphi(s) - \varphi(\mu))d\mu \right) ds .$$

Then we put $C_1(x)$ in the form

$$C_1(x) = \iint_{D_1} E_1(x,t,\tau)dt\,d\tau + \frac{1}{2}\iint_{D_2} \left(E_1(x,t,\tau) + E_1(x,\tau,t)\right)dt\,d\tau , \qquad (40)$$

where

$$D_1 : (x \leq t \leq \ell, 0 \leq \tau \leq x) , \; D_2 : (0 \leq t \leq x, 0 \leq \tau \leq x) .$$

Using (6) and lemma 2, one can prove that the integrand functions in both integrals (40) are non-negative in the corresponding regions. Hence, $C_1(x) \geq 0$ for $0 \leq x \leq \ell$ . It can be proved in analogous manner that $C_2(x) \geq 0$ for $0 \leq x \leq \ell$ . Then it follows from (27) and (39) that

$$I_3(x) \geq 0 \quad \text{for} \quad 0 \leq x \leq \ell . \qquad (41)$$

We obtain from (35), (38), (41), (30) that $r(x) \geq 0$ for $0 \leq x \leq \ell$ ( $\zeta(\ell,0) > 0$ according to lemma 1). This proves the theorem 2.

*Note 1.* Suppose that $\varphi(x) \equiv 0$ . Then the second inequality (27) becomes identity and the first inequality (27) turns into (13) (and at the same time becomes the necessary condition for the pointwise contact). As it has been shown above, the condition (13) is weaker in this case than the condition (15). Hence, new condition (27) is not stronger than the condition (15), which has been obtained in [3]. It is the merit of the sufficient condition.

*Note 2.* If $\varphi(x) = x/R$ , where $R \geq 2L/\pi$ then the inequalities (27) coincide.

## Conclusions

The uniqueness of the solution of two-leaf spring bending problem is proved. New sufficient condition, which is not stronger than one established before, is found for the pointwise contact between the spring leaves. These results widen the set of the leaf-spring and loading parameters for which the presence of the pointwise contact can be guaranteed. For the flat leaves the new condition is at the same time necessary. But it remains unknown whether this condition is necessary for the curved leaves. All the more, the general solution of the problem 1 (including the cases of the non-pointwise contact) is not obtained. This solution should be the subject of the further investigation.

## References

1. OSIPENKO M.A., NYASHIN Y.I., RUDAKOV R.N. A problem with the unilateral constraints in the theory of bending of foot prosthesis. **Russian Journal of Biomechanics**, 2(1-2): 11-18, 1998.
2. GEIL M.D., PARNIANPOUR M., BERME N. Significance of nonsagittal power terms in analysis of a dynamic elastic response prosthetic foot. **Journal of Biomechanical Engineering**, October, 121: 521-524, 1999.
3. OSIPENKO M.A., NYASHIN Y.I., RUDAKOV R.N. On the theory of bending of foot prosthesis containing the curved plates. **Russian Journal of Biomechanics**, 3(3): 73-77, 1999.
4. NYASHIN Y.I., OSIPENKO M.A., RUDAKOV R.N., SHIRYAEV M.I. Optimization of the foot prosthesis design parameters. Proceedings of the 6th International School on Biomechanics, Blagoevgrad, Bulgaria, 140-143, 1995.
5. RUDAKOV R.N., OSIPENKO M.A., NYASHIN Y.I., KALASHNIKOV Y.V., PODGAETZ A.R. Optimization and investigation of the foot prosthesis operating characteristics. **Russian Journal of Biomechanics**, 1(1-2): 1-11, 1997.

# ДОСТАТОЧНОЕ УСЛОВИЕ ТОЧЕЧНОГО КОНТАКТА ПРИ ИЗГИБЕ ДВУХЛИСТОВОГО ИСКРИВЛЕННОГО УПРУГОГО ЭЛЕМЕНТА ПРОТЕЗА СТОПЫ

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Доказана единственность решения задачи о совместном слабом изгибе искривленных листов двухлистовой рессоры, используемой в упругом элементе протеза стопы. Найдено новое, не являющееся более сильным, чем известное ранее, достаточное условие точечного контакта листов (когда профили листов соприкасаются, кроме точки защемления, только в одной точке). Полученные результаты вносят определенный вклад в еще не завершенную теорию изгиба листовых рессор, применяемых в протезостроении. Библ. 5.

Ключевые слова: протез стопы, упругий элемент, листовая рессора, искривленные листы, слабый изгиб, точечный контакт

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