Russian Journal of Biomechanics, Vol. 4, № 1: **-**, 2000

MATHEMATICAL DESCRIPTION OF INTERPOPULATIONAL INTERACTIONS PROCESSES

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Abstract: The article is devoted to mathematical modelling of processes which can be conditionally called "dynamic contact". The basic hypotheses corresponding to such processes are proposed. In this paper we devives the construction of different variants of the model of process like "dynamic contact". We provide the numerical results of those several cases.

Key words: dynamic contact, predator-prey, population, concentration, intensity

Introduction

There is a number of processes which can have analogous mathematical description because of similarity of the character of proceeding though having different physical nature.

We are interested in the process which can be conditionally called “dynamic contact”. The essence of this process consist in the following. Suppose that two evenly intermixed (in sense of probability) populations potentially mutually reacted with each offer are in continuous interaction. It is considered that the process is not momentary but dynamically and intensity of the contact depends on the concentrations of the interacted objects. We shall conditionally call these objects W and M. It is supposed that the products of reaction don’t influence the course of the process and conditions of its proceeding don’t change of time. We can give following examples of the process answering such a description: irreversible reaction between two chemical substances with mowing the products of reaction away; the process of impregnation of animals in the period of sexual activity (conformably to the population on the whole); the process of binding the antigen by immunocompetence cells assuming the lack of their reproduction and so an and so forth.

In biological systems similar processes frequently proceed simultaneously with reproductive processes or they are their necessary link. For instance, the condition of antigen non-reproduction and immunocompetence cells destroying them leads to the above-mentioned described process and this process initiates and stimulates the intensified growth of quantity the appropriate immunocompetence cells.

It is convenient to single out the similar processes into a separate category having analyzed and forecasted their properties.

Main peculiarities and regularities of processes like “dynamic contact”

In the capacity of the characteristics we shall consider the concentrations (quantity) of the opposite populations — M and W as function of time (essentially in that way the current state of the system will be described) and the rate of change of these characteristics (that reflect the intensity of the process). In other words the problem consists in the connections between $\dot{W}$, $\dot{M}$, W and M.

Consider the basic hypotheses corresponding to the processes like "dynamic contact".

1. Suppose that particle *W* can react with *c* the particles a sort of *M*. In its turn the particle *M* can react only with one particle *W*.
2. It is obvious that a number of contacts must be equal n both sides but as *W* particles are spent *c* times less the speed of expense must be *c* times less than among *M* particles.
3. It is evident that *M* and *W* are monotonically decreasing functions of time. One may notice that in the end (at $t \to \infty$) the mixture or (if the initial concentrations *M* and *W* properly selected) will completely react, or (if there is a surplus of one of them) the redundant matter will remain in residuum. Moreover it may be instantly assumed proceeding from physical sense that the reaction will pass without residuum with $M_0 = c\ W_0$, where M_0 and W_0 — are the initial concentrations.
4. It is convenient to place for consideration the number of the observed contacts *S(t)*. It is evident that *S(t)* is a temporal monotonically increasing function of time, limited above either M_0 or cW_0 (in general case $S(t)=M(t) - M_0 = c(W(t)-W_0)$).
5. Proceeding from physical sense one may introduce some general hypotheses for quantity and rate *S(t)* depending an current concentrations *M* and *W*. Consider the situation on the side of *W* population. Every *W* particle can potentially react with any *M* particle and probability of contact must grow if the amount of *M* increases and it must fall if the compactness of *W* particles rises (by reason of increase of competition). However the very number of contacts in a unit of time, i.e. rate *S(t)* must obligatory diminish with lowering of the amount g as *M* particles, as *W* particles.

The process when one *W* particle contacts with *c* particles like *M* is complicated for description just because it is necessary to examine its such features as possible change of properties of *W* particles after the first and next contacts. Besides the number of contacts of one *W* particle is not obligatory fixed and may be random variable. In order to expose characteristic peculiarities of the processes such as "dynamic contact" we shall choose and consider the process when one *W* particle contacts with one *M* particle and the formed reaction product removed and does not influence the subsequent proceeding of the reaction.

Pass on to the direct examination of different mathematical models. Let's mention that the processes similar to traditionally described systems "predator-prey" quite do for the description of the process "dynamic contact". The main pint of the latter ones in the fact that one population conditionally called "predator" feeds on "prey" and increased owing to it; the second one — so-called "prey" would steadily propagate itself if the "predator" does not eat it up. The likeness is that in both cases the dynamics of the process is determined by the intensity of the "contact" which means "eating up" in that case. We may say that in the processes like "dynamic contact" both populations are "predator" (and "prey" at the same time) with respect to each other.

It is clear from physical sense that the intensity of the contact is the higher the more quantity as one population as another one, and in the present situation it is determined by current population magnitudes. Hence one of the methods of approach to the process modelling is the following:

$$\frac{dW}{dt} = f_W(M,W),$$
$$\frac{dM}{dt} = f_M(M,W),$$

where $f(M,W)$ reflects the dependence of the process intensity on the concentrations of the contacting populations.

Below we shall consider correlations concerning only *W* as the correlations relative to *M* must be obviously similar.

It is evident that $f(M,W)$ must obey two principal conditions:

1. $f(M,W) \leq 0, \forall M, \forall W$ — this is connected with steady lowering of the number of *W* and *M* particles in a result of the contact.
2. $\frac{\partial f}{\partial M} > 0; \frac{\partial f}{\partial W} > 0$ — it reflect the positive connection of the reaction's speed with the quantity of contacting populations.

We shall examine the easiest approach to the description of processes like "predator-prey", using by us to the process like "dynamic contact". The process intensity (or the expense's speed of the contacting particles or the contact's probability) is directly accepted proportional to quantity (concentrations) of the contacting particles:

$$\dot{W} = -kMW . \quad (1)$$

Such correlations were often used when constructing mathematical models in immunology when modelling infections diseases [1-8]. The infectious agents interventioning in the organism (viruses, bacteria etc.) were essentially considered in the role of "prey" and the cells of the immune system destroying them were considered in the role of "predator". In order to connect the contact's intensity with current concentrations of the infectious agents and immunocompetent cells correlations in form (1) were used that allowed to get qualitative decisions.

However such an approach to description of the processes like "dynamic contact" can't be considered satisfactory. It is connected with the fact that this approach doesn't take into account the correlation of quantity of the contacting populations which can highly cardinally change the whole process. That is why such an approach can be used rather in those cases when quantity correlation of contacting populations is similar to the optimum and is relatively constant in the curse of process.

We offer such an approach which takes into consideration the correlations of quantity of contacting populations an condition of implementation of some empirical hypotheses.

For simplicity consider the situation when the reacted substances are utterly spent. In this case the equations describing the expense must be not only analogous but also identical. Therefore let us examine the construction of correlations concerning only *W*.

We consider that the intensity of the process (or the speed of expenditure of contacting particles or probability of the contact) is proportional to the concentration W with variable coefficient of proportionality depending on the correlation of concentrations. Thus we may note:

$$\begin{gathered} \dot{W} = -p_W(M,W)W, \\ p_W \geq 0,\ t \in (t_0, t_0 + \Delta t). \end{gathered} \quad (2)$$

Below we shall often use the following sort of correlations:

$$\begin{gathered} \dot{W} = -kp(M,W)W, \\ 0 \leq p \leq 1,\ t \in (t_0, t_0 + \Delta t). \end{gathered} \quad (2.2)$$

Coefficient *k* expresses the degree of reaction proceeding intensity on the whole. In this case relative rate of expense $\frac{\dot{W}}{W}$ is limited above.

Function *p* expressing the correlation of the population quantity possesses certain virtues which are necessary to be examined.

Function *p(M,W)* characteristics

1. Function *p(M,W)* must reflect the fact that for every individual *W* particle the rest *W* particles are competitors for the contact with potentially contacting *M*-particles. Therefore the

contact probability of every individual particle W rises with the increase of the amount of particles M and lower with the increase of the amount of competing particles W. Hence the first characteristic following:

$$\frac{\partial p}{\partial M} > 0 , \frac{\partial p}{\partial W} < 0 .$$

2. Taking into account the irreversible character of the process when the number of reagents can only decrease as a result of the contact, one may affirm that $p(M,W) \geq 0$. $p(M,W)=0$ means that particles M necessary for a contact come to an end, that is $M=0$.

Both of the described characteristics concerned the function $p_W(M,W)$ essentially expressing the relative rate $\frac{\dot{W}}{W}$. It is evident that in conformity with population of particles M the reasoning will be analogous, i.e. the function $p_M(W,M)$ expressing the relative rate $\frac{\dot{M}}{M}$ have the analogous characteristics.

3. It is obvious that the removal of correlation of population quantity to one or anther side must be expressed in change of functions $p_W(M,W)$ and $p_M(W,M)$, and one of them will increase and the second one will decrease. Suppose that exists functional dependence between functions $p_W(M,W)$ and $p_M(W,M)$. Then we may maintain that $\frac{dp_W}{dp_M} < 0$.

Thus within the bounds of the given approach we need an exact mathematical and physically grounded expression of interdependence of functions $p_W(M,W)$ and $p_M(W,M)$ for construction of the model.

Construction of different variants of the model of process like "dynamic contact"

We shall examine the equations describing the process in form (2.2):

$$\dot{W} = -kp_W(M,W)W ,$$

$$\dot{M} = -kp_M(W,M)M .$$

Since in the process of contact the expense correlation of the particles of both populations remains constant then if one accept it equal for simplicity we shall obtain:

$$-kp_W(M,W)W = -kp_M(W,M)M ;$$

Whence:

$$\frac{P_W}{P_M} = \frac{M}{W} . \tag{3}$$

A concrete mathematical expression depends on choice of the form of intercommunication between $p_W(M,W)$ and $p_M(W,M)$. We shall examine some particular cases:

1. Suppose that the sum p_M and p_W is constant. Taking into account an arbitrary choice of the coefficient k we shall consider that

$$p_W + p_M = 1;$$

consequently –

$$\frac{P_W}{1 - P_W} = \frac{M}{W}$$

In that case

$$P_W = \frac{\frac{M}{W}}{1+\frac{M}{W}} = \frac{M}{W+M}, \qquad P_M = \frac{\frac{W}{M}}{1+\frac{W}{M}} = \frac{W}{W+M}. \tag{4}$$

finally we shall receive:

$$\dot{W} = -k\frac{MW}{W+M}, \qquad \dot{M} = -k\frac{MW}{W+M}. \tag{5}$$

The obtained equations:

a) are resembling the equations like “predator-prey” (1) (the difference is in division of the right part by total quantity of populations $M+W$ taking part in the process);

b) the assumption $p_W + p_M = 1$ leads to the fact that the rate of diminution of the product MW becomes proportional to the product of the populations MW quantities:

$$\frac{d}{dt}MW = -kMW.$$

We shall note that using the equations like “predator-prey” (1) it turns out that the rate of diminution of the product MW is proportional to the product of the quantities of populations MW and of the sum $M+W$:

$$\frac{d}{dt}MW = -kMW(M+W).$$

c) It is interesting to compare the dynamics of the process. In case of using the system of equations (5) the sum of relative rates is constant –

$\frac{\dot{W}}{W} + \frac{\dot{M}}{M} = -k$, and in the event of using equations like “predator-prey” (1) it is proportional to the sum $M+W$ –

$\frac{\dot{W}}{W} + \frac{\dot{M}}{M} = -k(M+W)$, i.e. eventually it monotonously reduces.

d) It is convenient to consider such a characteristic of the process assumption population’s quantity ratio $S = \frac{M}{W}$. As formulas (4) indicate the functions p_M and p_W can be expressed through this parameter or inverse $S' = \frac{W}{M}$:

$$p_W(S) = \frac{S}{1+S}, \qquad p_M(S') = \frac{S'}{1+S'}.$$

We shall examine more general case when particles M are spent c times more than particles W, i.e. $\dot{M} = c\dot{W}$. When doing this

$$\frac{P_W}{1-P_W} = \frac{M}{cW} \Rightarrow$$

$$P_W = \frac{M}{cW+M}, \qquad P_M = \frac{cW}{cW+M}. \tag{6}$$

$$\dot{W} = -k\frac{MW}{cW+M}, \qquad \dot{W} = -kc\frac{MW}{cW+M}. \tag{7}$$

If we bring in the variables $U = MW$, $V = cW + M$ with the initial values $U_0 = M_0W_0$, $V_0 = cW_0 + M_0$, we shall get the system of equations

$$\dot{U} = -kU, \quad \dot{V} = -\frac{2ckU}{V}. \tag{8}$$

Solving it we get the following analytical conclusions:

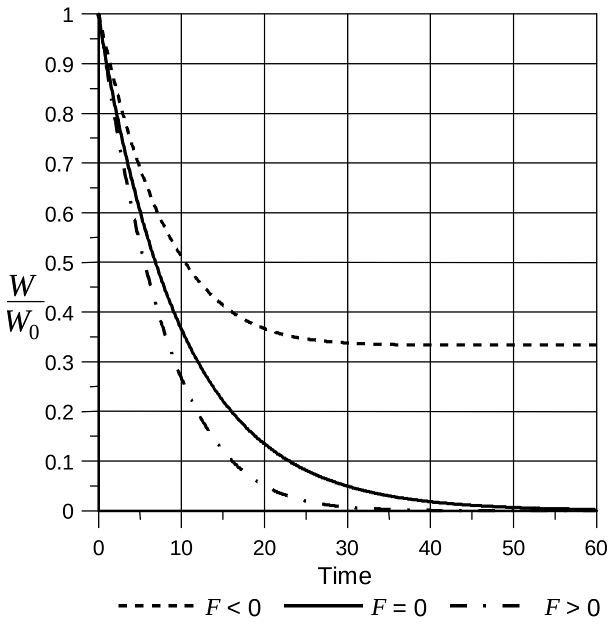

Fig. 1. Typical solutions of the equations' system (7)

the product of concentrations is proportional to the negative exponent of time – $U(t)=U_0e^{-kt}$, and their sum $V(t)=\sqrt{4cU+F^2}$, where $F=M_0-cW_0$.

It is possible to show that

$$M(t)=\frac{F}{2}+\sqrt{\left(\frac{F}{2}\right)^2+cU_0e^{-kt}} \quad , \quad W(t)=-\frac{F}{2c}+\sqrt{\left(\frac{F}{2c}\right)^2+\frac{1}{c}U_0e^{-kt}} .$$

Typical solutions of the equations' system (7) are represented in Fig. 1.

Fig. 1 shows three possible variants of the process proceeding:

a) in case $F=0$ (solid line) the initial concentrations of reagents are such that the mixture reacts completely; the diagram for $M(t)$ has analogy with the represented diagram of $W(t)$;

b) in case $F<0$ (dash line) substance W is in plenty and in the end of the process it remains in number of $-\frac{F}{c}=W_0-\frac{M_0}{c}$, whereas substance M is completely spent;

c) the case $F>0$ (dash dot line) is opposite the previous one. Now substance M is in plenty and in the end of the process it remains in number of $F=M_0-cW_0$. We may say that the substances exchange their places therefore the function W diagram in case b) may be considered as the function M diagram in that case and vice versa.

2. Assume that the product p_W and p_M is a constant. Taking into consideration an arbitrary choice of the coefficient k we shall consider that $p_Wp_M=1$; equation (3) transforms into

$$P_W{}^2=\frac{M}{W} \quad \Rightarrow$$

$$P_W=\sqrt{\frac{M}{W}}, \qquad P_M=\sqrt{\frac{W}{M}} .$$

The equations become the following:

$$\dot{W}=-k\sqrt{MW} , \qquad \dot{M}=-k\sqrt{MW} . \tag{9}$$

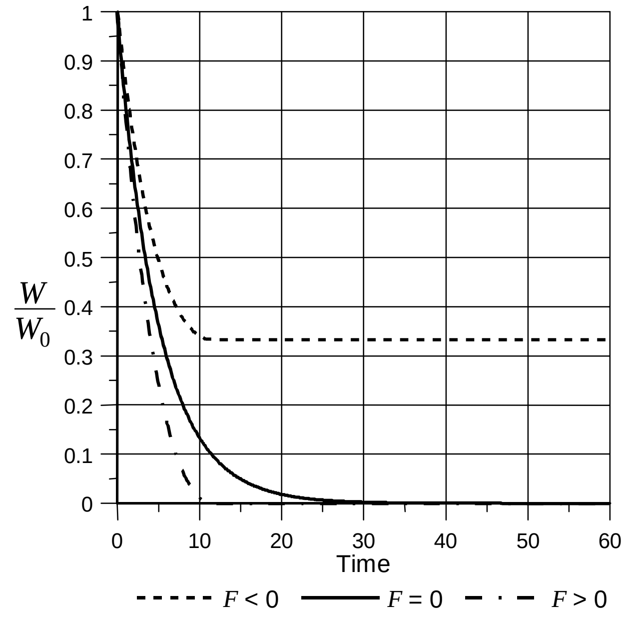

Fig. 2. Typical solutions of the equations' system (10)

In more general case when particles *M* are spent *c* times more than particles *W*, i.e. $\dot{M} = c\dot{W}$, we shall obtain:

$$P_W^{\,2} = \frac{M}{cW} \qquad \Rightarrow$$

$$P_W = \sqrt{\frac{M}{cW}}, \qquad P_M = \sqrt{\frac{cW}{M}}.$$

The equations become the following:

$$\dot{W} = -k\sqrt{\frac{1}{c}MW}, \qquad \dot{M} = -k\sqrt{cMW}. \tag{10}$$

Typical solutions of the equations' system (10) are represented in Fig. 2.

Interpretation of solutions in Fig. 2 (and in all subsequent figures) is analogous to the previous one in Fig. 1.

3. The next form of interdependence between p_W and p_M can be combined of two previous ones assuming preservation either of their sum or product.

Assume a certain synthesized intercommunication:

$$p_W + p_W p_M + p_M = 1.$$

Expressing P_M through p_W we shall get:

$$p_M = \frac{1 - p_W}{1 + p_W}. \tag{11}$$

Right now let's examine the case when particles *M* are spent *c* times more than particles *W*, i.e. $\dot{M} = c\dot{W}$. In such a case

$$\frac{P_W}{P_M} = \frac{M}{cW}.$$

As stated above we already inserted such a process characteristic as a ratio of population number $S = \frac{M}{W}$. In general case it is more convenient to examine the parameter $S = \frac{M}{cW}$. Its physical sense consists not so much in ratio of population number as how the offered on both sides possibilities for a contact are correlated with each offer (on the side of population M are offered M contacts, and on the side of population W are offered cW contacts). Generalizing we shall get:

$$\frac{P_W}{P_M} = S\ . \tag{12}$$

Taking into account (11):

$$\frac{P_W(1 + p_W)}{1 - P_W} = S\ .$$

Solving a certain quadratic equation concerning P_W we get two solutions: one of them is wittingly less than zero if values of parameter S are positive and doesn't meet physical limitations. The second solution on the contrary is strictly positive and it is an unknown quantity:

$$P_W = -\frac{1+S}{2} + \sqrt{\left(\frac{1+S}{2}\right)^2 + S}\ . \tag{13}$$

Drawing a conclusion $P_M(S)$ we shall mention that if it is transformed (12) to a form

$$\frac{P_M}{P_W} = \frac{1}{S} = \tilde{S}\ , \tag{14}$$

then function $P_M(\tilde{S})$ must be identical $P_W(S)$. parameter $\tilde{S}$ in physical sense is analogous to the parameter S opposite to it. In that way the functional dependence P_M on $\tilde{S}$ is analogous (13):

$$P_M = -\frac{1+\tilde{S}}{2} + \sqrt{\left(\frac{1+\tilde{S}}{2}\right)^2 + \tilde{S}}\ .$$

Transforming we shall obtain:

$$P_M = -\frac{1+S}{2S} + \frac{1}{S}\sqrt{\left(\frac{1+S}{2}\right)^2 + S}$$

Expressing p_M and p_W through M and W:

$$P_W = -\frac{M+cW}{2cW} + \frac{1}{cW}\sqrt{\left(\frac{M+cW}{2}\right)^2 + cMW}\ ,$$

$$P_M = -\frac{M+cW}{2M} + \frac{1}{M}\sqrt{\left(\frac{M+cW}{2}\right)^2 + cMW}\ . \tag{15}$$

The system of equations assume an air:

$$\dot{W} = k\left(\frac{M+cW}{2c} - \frac{1}{c}\sqrt{\left(\frac{M+cW}{2}\right)^2 + cMW}\right),$$

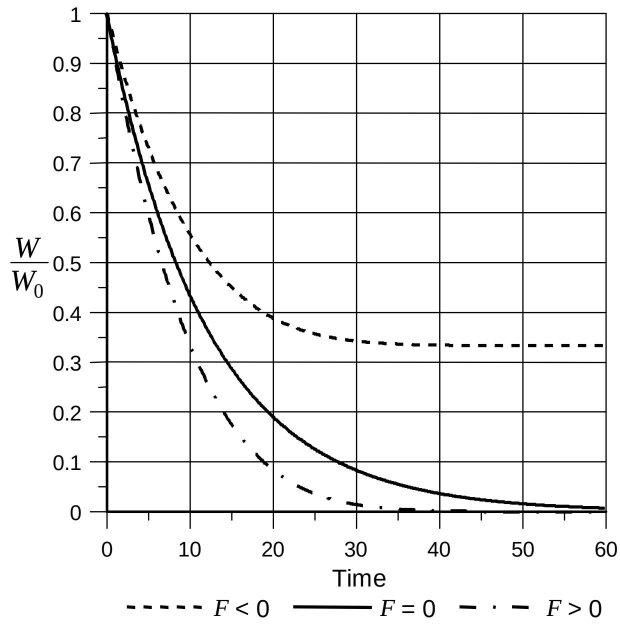

Fig. 3. Typical solutions of the equations' system(16).

$$\dot{M} = k\left(\frac{M + cW}{2} - \sqrt{\left(\frac{M + cW}{2}\right)^2 + cMW}\right). \qquad (16)$$

Typical solutions of the equations' system (16) are represented in Fig. 3.

4. We shall summarize the first examined variant of intercommunication between p_W and p_M. Suppose that the following is being fulfilled:

$$p_W{}^{\alpha} + p_M{}^{\alpha} = 1, \qquad (17)$$

where the index of power $\alpha > 0$.

We shall immediately consider a generalized variant when particles *M* are spent *c* times more than particles *W*, i.e. $\dot{M} = c\dot{W}$. Then the correlation

$\frac{P_W}{P_M} = \frac{M}{cW}$ taking account of (17) transforms into

$$\frac{p_W{}^{\alpha}}{1 - p_W{}^{\alpha}} = \frac{M^{\alpha}}{c^{\alpha}W^{\alpha}}.$$

Working out the last correlation as an equation concerning p_W, we shall get

$$p_W = \frac{M}{\sqrt[\alpha]{M^{\alpha} + c^{\alpha}W^{\alpha}}}. \qquad (18.1)$$

Allowing for (17)

$$p_M = \frac{cW}{\sqrt[\alpha]{M^{\alpha} + c^{\alpha}W^{\alpha}}}. \qquad (18.2)$$

We take notice that in case $\alpha = 1$ we get afore examined variant (dependence of the form (6) and equations (7)).

We shall examine another two possible variants:

a). $\alpha = 2$. Then

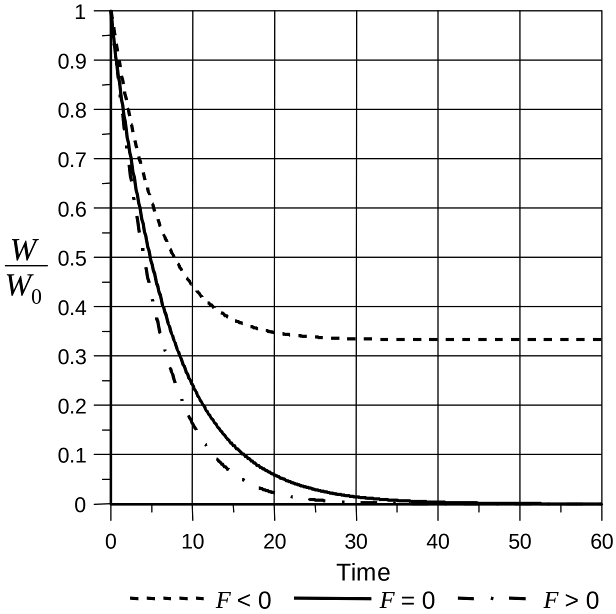

Fig. 4. Typical solutions of the equations' system (19) in case $\alpha = 2$

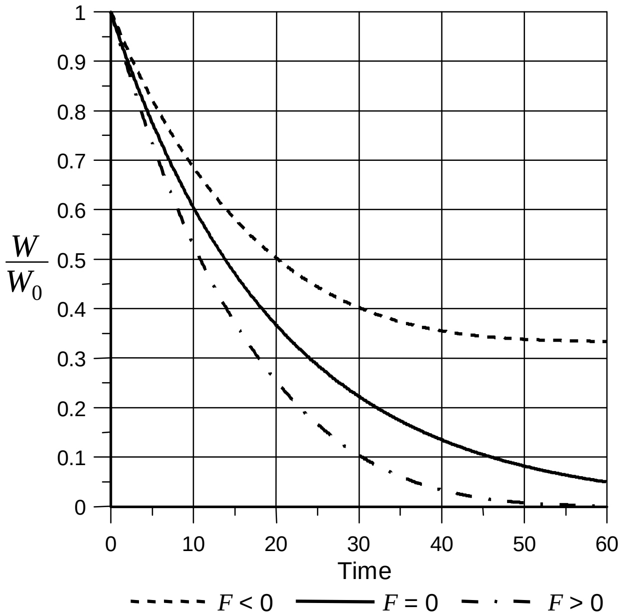

Fig. 5. Typical solutions of the equations' system (19) in case $\alpha = \frac{1}{2}$.

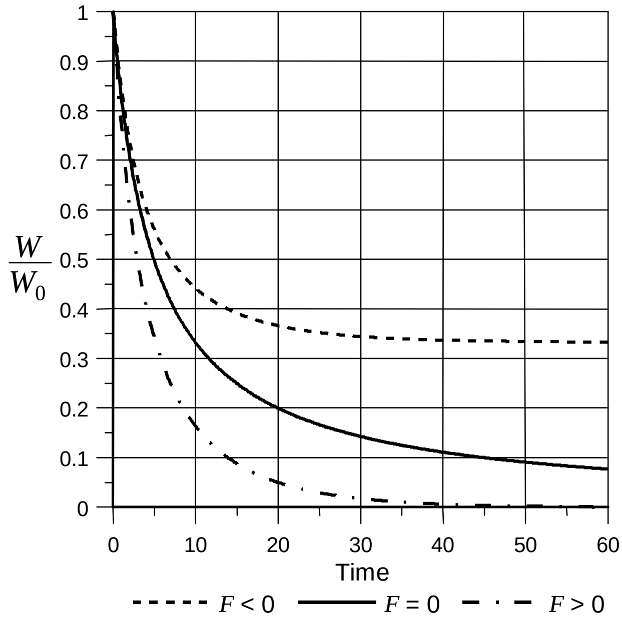

Fig. 6. Typical solutions of the equations' system (1)

$$p_W = \frac{M}{\sqrt[2]{M^2 + c^2W^2}}, \qquad p_M = \frac{cW}{\sqrt[2]{M^2 + c^2W^2}}.$$

Typical solutions in the present case are represented in Fig. 4.

b) $\alpha = \frac{1}{2}$. Then

$$P_W = \frac{M}{\left(\sqrt{M} + \sqrt{cW}\right)^2} = \frac{M}{M + 2\sqrt{cMW} + cW},$$

$$P_M = \frac{cW}{\left(\sqrt{M} + \sqrt{cW}\right)^2} = \frac{cW}{M + 2\sqrt{cMW} + cW}.$$

In this case typical solutions are represented in Fig. 5:

In the general case the equations look in such a way:

$$\dot{W} = -k\frac{MW}{\sqrt[\alpha]{M^\alpha + c^\alpha W^\alpha}}, \qquad \dot{M} = -k\frac{cMW}{\sqrt[\alpha]{M^\alpha + c^\alpha W^\alpha}}. \tag{19}$$

For comparison there are typical solutions of the equations' system like "predator-prey" (1) in Fig. 6:

Conclusion

As is clear from the diagrams all the solutions demonstrate monotonous lowering of quantity of populations until one of them is not exhausted. This is determined by difference of the initial values $F = M_0 - cW_0$. When correlation of quantities is optimum ($F = 0$) both populations are completely worn out in time. However dynamics of processes is considerably various if the form of choice of intercommunication between p_W and p_M is different. The choice of intercommunication corresponding to the real process – it is a separate major problem.

References

1. АЛЕКСЕЕВ А.С., БОНДАРЕНКО Н.П. Простейшая математическая модель процессов профилактики и лечения туберкулеза. **Динамика биологических систем,** 1: 109-117, 1977 (in Russia).
2. БАЛАНТЕР Б.И. , ХАНИН М.А. , ЧЕРНАВСКИЙ Д.С**. Введение в математическое моделирование патологических процессов**, Медицина, Москва, 1980 (in Russia).
3. БЕЛЫХ Л.Н. **Анализ математических моделей в иммунологии**, Наука, Москва, 1988 (in Russia).
4. ДИБРОВ Б.Ф. , ЛИФШИЦ Н.А. , ВОЛЬКЕНШТЕЙН М.В. Математическая модель иммунной реакции. III. Описание инфекционного процесса с учетом изменения численности В-лимфоцитов. **Биофизика**, 23(1): 143-147, 1978 (in Russia).
5. МАРЧУК Г.И. **Математические модели в иммунологии. Вычислительные методы и эксперименты**, Наука, Москва, 1991 (in Russia).
6. МАРЧУК Г.И., БЕЛЫХ Л.Н. **Математические модели в иммунологии и медицине**, Наука, Москва, 1986 (in Russia).
7. МОЛЧАНОВ А.М., НАЗАРЕНКО В.Г., ШАТУРСКИЙ И.Г. Анализ модели однобарьерного иммунитета. **Биофизика**, 16(4): 143-147, 1971 (in Russia).
8. РОМАНОВСКИЙ Ю.М., СТЕПАНОВА Н.В., ЧЕРНАВСКИЙ Д.С. **Математическая биофизика,** Наука, Москва, 1984 (in Russia).

МАТЕМАТИЧЕСКОЕ ОПИСАНИЕ ПРОЦЕССОВ МЕЖПОПУЛЯЦИОННЫХ ВЗАИМОДЕЙСТВИЙ

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Существует целый ряд процессов, которые при различной физической природе в силу схожести характера протекания могут иметь аналогичное математическое описание. Нас интересует процесс, который условно можно назвать "динамический контакт". Сущность этого процесса заключается в следующем. Предположим, что в непрерывном взаимодействии находятся две равномерно перемешанные (в вероятностном смысле) популяции, потенциально взаимореагирующие друг с другом. Считается, что процесс не мгновенный, а динамический, и интенсивность контакта зависит от концентраций взаимодействующих объектов. Предполагается, что продукты реакции не влияют на течение процесса и условия его протекания не меняются во времени. В качестве характеристик процесса рассматривались концентрации взаимодействующих популяций как функции от времени и их скорости изменения. Рассмотрены основные особенности и закономерности данных процессов. Построены и проанализированы различные математические модели, описывающие процессы типа "динамический контакт".

Ключевые слова: динамический контакт, хищник-жертва, популяция, концентрация, интенсивность.

Received 15 March 2000