Russian Journal of Biomechanics, № 1: **-**, 1999

CENTER OF RESISTANCE AND CENTER OF ROTATION OF A TOOTH: THE DEFINITIONS, CONDITIONS OF EXISTENCE, PROPERTIES

M. A. Osipenko, M.Y. Nyashin, Y.I. Nyashin

Department of Theoretical Mechanics, Perm State Technical University, 29a, Komsomolsky Prospect, 614600, Perm, Russia

Abstract: In the present study, the exact mathematical definitions of concepts of a “center of resistance” and a ”center of rotation of a tooth” are introduced. These concepts are widely used in the literature on orthodontic biomechanics, however they are nonstrictly defined. A center of resistance and a center of rotation are shown to exist only in particular cases. The conditions of their existence and formulas to determine their positions are given.

Key words: dental biomechanics, orthodontics, analytical mechanics, center of resistance, center of rotation

Introduction

In dental biomechanics, there are concepts to be discussed. In particular, two biomechanical concepts - a “center of resistance” (CRE) and a “center of rotation of a tooth” (CRO) were used in a series of experimental and theoretical investigations [1-17] appeared for the last three decades. These concepts are introduced in studies of tooth movement under the action of both short-term forces when after unloading all the displacements are supposed to vanish and tooth recovers its origin location and long-term forces causing alveolus remodelling. For example, these concepts are useful to determine forces which should cause controlled orthodontic tooth movement [10]. However the various authors differently treat these concepts for lack of their exact mathematical definitions.

Burstone and Pryputniewicz [2] have defined the CRE as a point where a single force for which the line of action passing through the CRE produces tooth translation in the direction of the line of action of the applied force. The similar definitions were used in a number of other works [4,7,13,14]. Moreover Hocevar [7] has underlined, that the couple consisting of two forces of equal magnitude acting in parallel but opposite direction and having different points of application induces tooth rotation about the CRE.

Hocevar [7], Demange [4] have noticed, that for a restrained body, what the tooth is, the CRE is similar to the center of mass of a free body. The position of the CRE depends not only on geometry and properties of the tooth itself but also on its environment, i.e. periodontal ligament, cortical and spongy bones, adjacent teeth. As Demange [4] has considered, a position of the CRE is a function of anatomy of a tooth, the nature of the periodontal structures, density of the alveolar bone, the elasticity of the desmodontal structures, patient’s age. Therefore it is more correct to say [16] a “center of resistance associated with the tooth” instead of a “center of resistance of the tooth”.

The concept of a CRO was also used in various senses. For example, Christiansen and Burstone [3], Hurd and Nikolai [8] have stated that the CRO is always somewhere along the line representing the long axis of the tooth, whereas Smith and Burstone [14] have stated that this need not be so.

According to Demange [4] the position of the CRO depends exclusively on the system of forces applied. A distance from the CRO to the CRE determines the type of movement obtained. If the CRO and the CRE are identical, then the movement is pure rotation. If this distance tends to infinity, then translation, or bodily movement, occurs. Between these two extreme situations, combined effects of rototranslation may occur, i.e. an arbitrary force system tends to tooth movement in which the CRE moves along a straight line and the tooth rotates about the CRE simultaneously.

In some studies, concepts of the CRE and the CRO were used to analyse only the special cases, in which the oversimplifying assumptions were made, such as the anatomy of the root was represented by idealized geometric form (parabola [3], circular cone [13]); the single-rooted teeth were considered [3,13,14,15], periodontal ligament was assumed to be homogeneous, isotropic material [13,15], the studies were limited by two-dimensional representations [12,13].

In the present paper, the exact mathematical definitions of concepts of the CRE and the CRO for a tooth having an arbitrary spatial geometry and a mass distribution and inhomogeneous, nonisotropic periodontal ligament are introduced.

Tooth movement in the alveolar socket is considered assuming that resorption and apposition of the alveolar bone do not occur. This type of displacement was called “primary” by Burstone and Pryputniewicz [2]; Smith [14] have defined it as “instantaneous”.

Model of a loaded tooth immersed in periodontal ligament

Tooth is assumed to be a rigid body, as the Young’s modulus of a tooth is approximately 100,000 times that of a periodontal ligament [17]. Let O*X*Y*Z* be the fixed reference system rigidly connected with the alveolar bone and OXYZ be the reference system rigidly connected with the tooth, these systems coinciding when any force acting on the tooth is absent. Under a load the tooth moves. According to the six degrees of freedom of the motion of the rigid body, consider six generalized coordinates of the tooth: three coordinates ξ, η, ζ of the point O in the reference system O*X*Y*Z* ($\xi=\eta=\zeta=0$ without load) and three parameters assigning the orientation of OXYZ with respect to $O\tilde{X}\tilde{Y}\tilde{Z}$, where axes $\tilde{X}, \tilde{Y}, \tilde{Z}$ are parallel to axes X*, Y*, Z*, respectively. It is convenient to select these parameters as follows. It is known [18] that any motion of a rigid body with one fixed point can be realized by the rotation of the rigid body about straight line passing through this point. Let **n** be the unit vector on the axis of the rotation transforming $O\tilde{X}\tilde{Y}\tilde{Z}$ into OXYZ; θ be a rotation angle. Direction of the rotation and direction of vector **n** are co-ordinated by the right-hand screw rule. It is assumed that $0 \le \theta \le \pi$ to avoid lack of uniqueness in definition of **n** and θ, for simultaneous changes of variables $\mathbf{n} \to \mathbf{-n}$, $\theta \to 2\pi - \theta$ result in the same rotation. Let the components of **n** in O*X*Y*Z* (or $O\tilde{X}\tilde{Y}\tilde{Z}$) be denoted n_x^*, n_y^*, n_z^*. Then the above-mentioned parameters assigning the orientation are as follows:

$$\lambda = 2n_x^* \sin\tfrac{\theta}{2},\ \mu = 2n_y^* \sin\tfrac{\theta}{2},\ \nu = 2n_z^* \sin\tfrac{\theta}{2}.$$

Here λ, μ, ν are notations of doubled Euler parameters ($\lambda=\mu=\nu=0$ without load, for in this case **n** is uncertain, but $\theta=0$) [19].

Thus generalized coordinates of the tooth are ξ, η, ζ, λ, μ, ν, which are grouped into two columns:

$$\boldsymbol{\rho} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix},\ \boldsymbol{\varphi} = \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}.$$

The orientation of OXYZ with respect to $O\tilde{X}\tilde{Y}\tilde{Z}$ may also be described by the orthogonal matrix, which is introduced as follows: if some point connected rigidly with the tooth has coordinate column **r** in OXYZ, then this point has coordinate column $\hat{V}$ **r** in $O\tilde{X}\tilde{Y}\tilde{Z}$ [19]. From this it follows that the coordinate column of the point in O*X*Y*Z* is **r***=**ρ**+ $\hat{V}$ **r**. Then a difference **u***=**r***-**r** (as a function of **r**) is the coordinate column of displacements of tooth points with respect to O*X*Y*Z*:

$$\mathbf{u}^*(\mathbf{r})=\boldsymbol{\rho}+\hat{V}\,\mathbf{r}-\mathbf{r}. \tag{1}$$

The coordinate column of displacements of tooth points with respect to O*X*Y*Z* is:

$$\mathbf{u}(\mathbf{r})=\hat{V}^T\,\mathbf{u}^*(\mathbf{r})=\hat{V}^T\,\boldsymbol{\rho}+\mathbf{r}-\hat{V}^T\,\mathbf{r}. \tag{2}$$

Elements of matrix $\hat{V}$ may be expressed in terms of λ, μ, ν as follows [19]:

$$\hat{V}=\frac{1}{2}\begin{pmatrix} 2-\mu^2-\nu^2 & \lambda\mu-\sigma\nu & \lambda\nu+\sigma\mu \\ \lambda\mu+\sigma\nu & 2-\lambda^2-\nu^2 & \mu\nu-\sigma\lambda \\ \lambda\nu-\sigma\mu & \mu\nu+\sigma\lambda & 2-\lambda^2-\mu^2 \end{pmatrix}, \tag{3}$$

where $\sigma=\sqrt{4-\lambda^2-\mu^2-\nu^2}$.

The periodontal ligament, in which the tooth is immersed, will be considered as an elastic medium, which potential energy is the given function of the generalized coordinates of a tooth Π(**ρ**,**φ**).

When assigning a load acting on the tooth, it is essential that firstly, tooth location (**ρ**,**φ**) dependence of the load should be given and secondly, this dependence must satisfy the condition, whereby tooth displacement is determined by the load at **ρ**=0, **φ**=0. (Otherwise the system of the tooth + the periodontal ligament + some object developing a load will be under investigation rather than that of the tooth + the periodontal ligament by the certain load.)

The load is considered to be developed by some elastic system, which potential energy Π_{load}(**ρ**,**φ**) is given, thereby the first requirement mentioned above is met. The second requirement will be satisfied below. The equations defining the tooth position (**ρ**,**φ**) at state of equilibrium are [18]:

$$\begin{cases} \dfrac{\partial\Pi}{\partial\boldsymbol{\rho}}+\dfrac{\partial\Pi_{\text{load}}}{\partial\boldsymbol{\rho}}=0, \\ \dfrac{\partial\Pi}{\partial\boldsymbol{\varphi}}+\dfrac{\partial\Pi_{\text{load}}}{\partial\boldsymbol{\varphi}}=0. \end{cases} \tag{4}$$

Further discussion will cover only small displacements of a tooth. It means that only the values of the first-order approximation by (**ρ**,**φ**) will be considered in (1)-(4). First of all, it is readily shown that in this case $\hat{V}$ **f**=**f**+**φ**×**f** (for an arbitrary column **f**). Then the equation (2) yields:

$$\mathbf{u}(\mathbf{r})=\boldsymbol{\rho}+\boldsymbol{\varphi}\times\mathbf{r}. \tag{5}$$

Further, to ensure an indicated order of approximation in (4) it is required to leave the terms of the second-order approximation by (**ρ**, **φ**) in Π(**ρ**,**φ**) and Π_{load}(**ρ**,**φ**). Since (**ρ**, **φ**) are chosen so that **ρ**=0, **φ**=0 at state of equilibrium (without load), Π(**ρ**,**φ**) does not contain linear in (**ρ**, **φ**) terms and can be written as:

$$n(\,,\,)=\tfrac{1}{2}\Big((\hat{A})\cdot+(\hat{B})\cdot+2(\hat{D})\cdot\ \Big), \tag{6}$$

where $\hat{A}$, $\hat{B}$, $\hat{D}$ are the given matrices ($\hat{A}$ and $\hat{B}$ are symmetric; $\hat{D}$ is asymmetric in general case). Because of stable equilibrium of tooth in periodontal ligament, a quadratic form (6) is positive definite. Therefore the matrices $\hat{A}$ and $\hat{B}$ are also positive definite.

In general, Π_{load}(**ρ**,**φ**) may include linear in (**ρ**, **φ**) terms:

$$n_{load}(\,,\,) = -\mathbf{R}\cdot - \mathbf{M}\cdot + \tfrac{1}{2}\Big((\hat{a})\cdot + (\hat{b})\cdot + 2(\hat{d})\cdot\ \Big), \tag{7}$$

where $\mathbf{R}$, $\mathbf{M}$ are the given columns; $\hat{a}, \hat{b}, \hat{d}$ are the given matrices.

It should be emphasized that the columns $\mathbf{R}$ and $\mathbf{M}$ have the following physical meanings. It can be shown that they contain components of the resulting force and the resulting moment of the load about the point O at ρ=0, φ=0, i.e. when OXYZ and O*X*Y*Z* coincide (this statement does not depend on assumption that tooth displacements are small). In fact, let $\mathbf{F}_k$ mean force system applied to tooth at ρ=0, φ=0, and let δρ, δφ be the variations of ρ and φ, respectively. From (3) we obtain: if φ=0 then $\delta\hat{V}^T\mathbf{r}=-\delta\boldsymbol{\varphi}\times\mathbf{r}$. It follows from (2) and (3) that $\delta\mathbf{u}(\mathbf{r})=\delta\boldsymbol{\rho}+\delta\boldsymbol{\varphi}\times\mathbf{r}$ at ρ=0, φ=0 and, therefore, the virtual work of system of forces $\mathbf{F}_k$ is:

$$\sum_k \delta A_k = \sum_k \mathbf{F}_k\cdot\delta\mathbf{u}(\mathbf{r}_k) = (\sum_k \mathbf{F}_k)\cdot\delta\boldsymbol{\rho} + (\sum_k \mathbf{r}_k\times\mathbf{F}_k)\cdot\delta\boldsymbol{\varphi}\,. \tag{8}$$

On the other hand,

$$\sum_k \delta A_k = \left(-\frac{\partial\Pi_{load}}{\partial\boldsymbol{\rho}}\cdot\delta\boldsymbol{\rho} - \frac{\partial\Pi_{load}}{\partial\boldsymbol{\varphi}}\cdot\delta\boldsymbol{\varphi}\right)\Bigg|_{\rho=0,\varphi=0},$$

and (7) yields:

$$\sum_k \delta A_k = \mathbf{R}\cdot\delta\boldsymbol{\rho} + \mathbf{M}\cdot\delta\boldsymbol{\varphi}\,. \tag{9}$$

Finally, when (8) is compared with (9), we conclude that the statement concerning the physical meanings of columns $\mathbf{R}$ and $\mathbf{M}$ is proved.

Substituting the equations (6) and (7) into (4), we derive:

$$\begin{cases}(\hat{A}+\hat{a}) + (\hat{D}+\hat{d}) = \mathbf{R},\\ (\hat{D}+\hat{d})^T + (\hat{B}+\hat{b}) = \mathbf{M}.\end{cases}$$

Suppose that the matrices $\hat{a}$, $\hat{b}$, $\hat{d}$ are negligibly small in comparison with the matrices $\hat{A}$, $\hat{B}$, $\hat{D}$, respectively, thereby the second requirement imposed on load is satisfied. In a qualitative sense, it means that stiffness of the elastic object developing the load is much less than that of the periodontal ligament. Then the final equations defining the tooth position (ρ,φ) at state of equilibrium are:

$$\begin{cases}\hat{A} + \hat{D} = \mathbf{R},\\ \hat{D}^T + \hat{B} = \mathbf{M}.\end{cases} \tag{10}$$

By virtue of the fact that potential energy (6) is positive definite, the system (10) has the unique solution, which can be written as:

$$\begin{cases} = \hat{\alpha}\mathbf{R} + \hat{\gamma}\mathbf{M},\\ = \hat{\gamma}^T\mathbf{R} + \hat{\beta}\mathbf{M},\end{cases} \tag{11}$$

where $\hat{\alpha}$, $\hat{\beta}$ are the symmetric and positive definite matrices, $\hat{\gamma}$ is asymmetric in general case. Note that with use of (10) and (11) the matrices $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ may be expressed in terms of the matrices $\hat{A}, \hat{B}, \hat{D}$ and vice versa. Specifically, the following relations will be used below:

$$\hat{\beta}=(\hat{B}-\hat{D}^T\hat{A}^{-1}\hat{D})^{-1},\ \hat{\gamma}=-\hat{A}^{-1}\hat{D}\hat{\beta},\ \hat{A}=(\hat{\alpha}-\hat{\gamma}\hat{\beta}^{-1}\hat{\gamma}^T)^{-1}. \tag{12}$$

Substituting the expressions (11) into (5) yields:

$$\mathbf{u}(\mathbf{r})=(\hat{\alpha}-\hat{r}\hat{\gamma}^T)\mathbf{R}+(\hat{\gamma}-\hat{r}\hat{\beta})\mathbf{M}, \tag{13}$$

where antisymmetric matrix $\hat{r}$ is introduced in the following way: $\hat{r}\mathbf{f}=\mathbf{r}\times\mathbf{f}$ for an arbitrary column $\mathbf{f}$. The elements of matrix $\hat{r}$ are:

$$\hat{\mathrm{r}} = \begin{pmatrix} 0 & -\mathrm{z} & \mathrm{y} \\ \mathrm{z} & 0 & -\mathrm{x} \\ -\mathrm{y} & \mathrm{x} & 0 \end{pmatrix}, \text{ where } \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix} = \mathbf{r}.$$

Center of a resistance: the definition, existence, properties

Definition 1.

The center of resistance of a tooth and its environment (CRE) is defined as a point rigidly connected with the tooth and satisfying two following requirements:
a) if **R**=0, then $\mathbf{u}(\mathbf{r}_{\mathrm{CRE}})=0$ for an arbitrary **M** (i.e. if applied forces are reduced to a couple, then the tooth rotates about the CRE);
b) if $\mathbf{M}=\mathbf{r}_{\mathrm{CRE}}\times\mathbf{R}$, then $\boldsymbol{\varphi}=0$ for an arbitrary **R** (i.e. if applied forces are reduced to a resultant with a line of action passing through the CRE, then tooth translation occurs).

It follows from the Definition 1 that the position of the CRE in OXYZ does not depend on applied forces.

Theorem 1.

If $\hat{\gamma}\,\hat{\beta}^{-1}$ is antisymmetric matrix, then there exists the unique CRE such that $\hat{\mathrm{r}}_{CRE}=\hat{\gamma}\,\hat{\beta}^{-1}$, otherwise the CRE does not exist.

Proof.

a) It follows from (13) that if **R**=0, then $\mathbf{u}(\mathbf{r})=(\hat{\gamma}-\hat{r}\,\hat{\beta})\mathbf{M}$. A condition $\mathbf{u}(\mathbf{r}_{\mathrm{CRE}})=0$ for an arbitrary **M** is equivalent to $\hat{\gamma}-\hat{\mathrm{r}}_{CRE}\,\hat{\beta}=0$. Thus, as $\hat{\beta}$ is nonsingular matrix,

$$\hat{\mathrm{r}}_{CRE}=\hat{\gamma}\,\hat{\beta}^{-1}. \tag{14}$$

Since $\hat{\mathrm{r}}_{CRE}$ is antisymmetric matrix, the condition (14) is met if and only if $\hat{\gamma}\,\hat{\beta}^{-1}$ is antisymmetric one, otherwise the CRE does not exist.
b) It follows from (11) that if $\mathbf{M}=\mathbf{r}_{\mathrm{CRE}}\times\mathbf{R}$, then $\boldsymbol{\varphi}=(\hat{\gamma}^{T}+\hat{\beta}\,\hat{\mathrm{r}}_{CRE})\mathbf{R}$. A condition $\boldsymbol{\varphi}=0$ for an arbitrary **R** is equivalent to $\hat{\gamma}^{T}+\hat{\beta}\,\hat{\mathrm{r}}_{CRE}=0$. Thus, as $\hat{\beta}$ is nonsingular matrix,

$$\hat{\mathrm{r}}_{CRE}=-\hat{\beta}^{-1}\,\hat{\gamma}^{T}. \tag{15}$$

Since $(\hat{\beta}^{-1}\,\hat{\gamma}^{T})^{\mathrm{T}}=\hat{\gamma}\,\hat{\beta}^{-1}$ and matrix antisymmetry (or lack of matrix antisymmetry) is not changed after its transposition, then the condition (15) is met if and only if $\hat{\gamma}\,\hat{\beta}^{-1}$ is antisymmetric one. In this case, (15) leads to the same expression for $\hat{\mathrm{r}}_{\mathrm{CRE}}$ as (14) does. Otherwise the CRE does not exist.

Note.

From (12) we have $\hat{\gamma}\,\hat{\beta}^{-1}=-\hat{A}^{-1}\,\hat{D}$ and therefore corresponding change of matrix can be made in the Theorem 1 statement.

We see that the CRE exists only under rather special conditions. That is why the problem of existence of the CRE for a tooth + periodontal ligament system having different types of symmetry will be considered. In essence, only the symmetry of tooth root is important, because that of tooth crown is of no significance in this study.

We will say that the tooth + the periodontal ligament system is *symmetric about some plane* if the reflection of an arbitrary load about this plane leads to the reflection of displacements of the tooth points about this plane. Suppose that such a symmetry exists and choose OXYZ so that the point O belongs to the symmetry plane. In this case, reflection may be described by means of orthogonal symmetric matrix $\hat{W}$ in the following manner [19]: if $\mathbf{r}$ is a coordinate column of some point in OXYZ, then $\hat{W}\mathbf{r}$ is the coordinate column of the reflected point in OXYZ. Then ($\hat{W}\mathbf{R}$, -$\hat{W}\mathbf{M}$) is a reflected load ($\mathbf{R}$, $\mathbf{M}$). Note that sign change here is evident from the following identity:

$$(\hat{W}\mathbf{f})\times(\hat{W}\mathbf{g})=-\hat{W}(\mathbf{f}\times\mathbf{g}), \tag{16}$$

which is valid for arbitrary columns $\mathbf{f}$ and $\mathbf{g}$. The load ($\mathbf{R}$, $\mathbf{M}$) causes the tooth displacement field (13). Moreover according to (13) the reflected load causes the following tooth displacement field:

$$\mathbf{u}_1(\mathbf{r})=(\hat{\alpha}-\hat{r}\,\hat{\gamma}^T)\hat{W}\mathbf{R}-(\hat{\gamma}-\hat{r}\,\hat{\beta})\hat{W}\mathbf{M}. \tag{17}$$

Further, $\hat{W}\mathbf{u}(\hat{W}\mathbf{r})$ is the reflected tooth displacement field $\mathbf{u}(\mathbf{r})$. Substituting (13) into (17) and taking into consideration that if matrix $\hat{r}$ is related to column $\mathbf{r}$, then matrix $-\hat{W}\,\hat{r}\,\hat{W}$ is related to column $\hat{W}\mathbf{r}$ (it follows from (16)), we find the reflected tooth displacement field:

$$\mathbf{u}_2(\mathbf{r})=\hat{W}(\hat{\alpha}+\hat{W}\,\hat{r}\,\hat{W}\,\hat{\gamma}^T)\mathbf{R}+\hat{W}(\hat{\gamma}+\hat{W}\,\hat{r}\,\hat{W}\,\hat{\beta})\mathbf{M}. \tag{18}$$

A consequence of symmetry is the fact that $\mathbf{u}_1(\mathbf{r})=\mathbf{u}_2(\mathbf{r})$ for arbitrary columns $\mathbf{R}$, $\mathbf{M}$ and an arbitrary antisymmetric matrix $\hat{r}$. Then it is seen from (17) and (18) that it is true if and only if:

$$\hat{\alpha}\,\hat{W}=\hat{W}\,\hat{\alpha},\ \hat{\beta}\,\hat{W}=\hat{W}\,\hat{\beta},\ \hat{\gamma}\,\hat{W}=-\hat{W}\,\hat{\gamma}. \tag{19}$$

Let us select OXYZ so that the symmetry plane passes through an axis X, and ψ be an angle of symmetry plane with an axis Y. Then:

$$\hat{W}(\psi)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos 2\psi & \sin 2\psi\\ 0 & \sin 2\psi & -\cos 2\psi\end{pmatrix}.$$

Consider different types of symmetry.

1). The tooth + the periodontal ligament system has only one symmetry plane. Let this symmetry plane be coincident with plane XY. In this case ψ=0 and thus using (19) for $\hat{W}(0)$ we have:

$$\hat{\alpha}=\begin{pmatrix}\alpha_{11} & \alpha_{12} & 0\\ \alpha_{12} & \alpha_{22} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix},\ \hat{\beta}=\begin{pmatrix}\beta_{11} & \beta_{12} & 0\\ \beta_{12} & \beta_{22} & 0\\ 0 & 0 & \beta_{33}\end{pmatrix},\ \hat{\gamma}=\begin{pmatrix}0 & 0 & \gamma_{13}\\ 0 & 0 & \gamma_{23}\\ \gamma_{31} & \gamma_{32} & 0\end{pmatrix}. \tag{20}$$

It is easy to verify that $\hat{\gamma}\,\hat{\beta}^{-1}$ is not antisymmetric in general case. Therefore the presence of symmetry plane does not ensure existence of the CRE.

2). The tooth + the periodontal ligament system has two mutually perpendicular symmetry planes. Let the symmetry planes be coincident with XY and XZ. In this case ψ=0 and $\psi=\pi/2$, thus using (19) for $\hat{W}(0)$ and $\hat{W}(\pi/2)$ we have:

$$\hat{\alpha}=\begin{pmatrix}\alpha_{11} & 0 & 0\\ 0 & \alpha_{22} & 0\\ 0 & 0 & \alpha_{33}\end{pmatrix},\ \hat{\beta}=\begin{pmatrix}\beta_{11} & 0 & 0\\ 0 & \beta_{22} & 0\\ 0 & 0 & \beta_{33}\end{pmatrix},\ \hat{\gamma}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \gamma_{23}\\ 0 & \gamma_{32} & 0\end{pmatrix}.$$

It is easy to verify that $\hat{\gamma}\,\hat{\beta}^{-1}$ is also not antisymmetric in general case. Therefore the presence of two mutually perpendicular symmetry planes does not ensure existence of the CRE.

3). We will say that the tooth + the periodontal ligament system is *symmetric about some axis* if the system is symmetric about each plane passing through the axis. Suppose that such a symmetry exists and choose OXYZ so that the symmetry axis is coincident with axis X. In this case the conditions (19) must be met for each ψ. Then:

$$\hat{\alpha}=\begin{pmatrix}\alpha_{11} & 0 & 0\\ 0 & \alpha_{22} & 0\\ 0 & 0 & \alpha_{22}\end{pmatrix},\ \hat{\beta}=\begin{pmatrix}\beta_{11} & 0 & 0\\ 0 & \beta_{22} & 0\\ 0 & 0 & \beta_{22}\end{pmatrix},\ \hat{\gamma}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \gamma_{23}\\ 0 & -\gamma_{23} & 0\end{pmatrix}.$$

It is easy to verify that $\hat{\gamma}\,\hat{\beta}^{-1}$ is now antisymmetric. Therefore according to Theorem 1 there exists the CRE such that:

$$x_{CRE}=-\frac{\gamma_{23}}{\beta_{22}},\ y_{CRE}=z_{CRE}=0,$$

So the CRE belongs to the symmetry axis.

Thus it may be concluded that existence of the CRE is ensured by axial symmetry of the tooth + periodontal ligament system. It should be noted that such a type of symmetry does not often occur.

We shall return now to the case, when the tooth + the periodontal ligament system has the only symmetry plane (OXYZ is selected so that the symmetry plane coincides with the plane XY). In this case we introduce a concept of the CRE in a limited (but wide enough) sense to ensure CRE existence without additional requirement of symmetry.

Definition 2.

The center of resistance of a tooth and its environment in a symmetry plane (CRES) is defined as a point belonging to the symmetry plane and satisfying two requirements of Definition 1 only if columns **R** and **M** look like:

$$\mathbf{R}=\begin{pmatrix}R_x\\ R_y\\ 0\end{pmatrix},\ \mathbf{M}=\begin{pmatrix}0\\ 0\\ M_z\end{pmatrix}. \tag{21}$$

For example, **R** and **M** are of type (21) if the load belongs to the symmetry plane.

Theorem 2.

There exists the unique CRES and:

$$x_{CRES}=-\frac{\gamma_{23}}{\beta_{33}},\ y_{CRES}=\frac{\gamma_{13}}{\beta_{33}}. \tag{22}$$

Proof.

a) It follows from (13), (20), (21) that $u_z=0$,

$$\begin{pmatrix}u_x(x,y)\\ u_y(x,y)\end{pmatrix}=\begin{pmatrix}\alpha_{11}-\gamma_{13}y & \alpha_{12}-\gamma_{23}y\\ \alpha_{12}+\gamma_{13}x & \alpha_{22}+\gamma_{23}x\end{pmatrix}\begin{pmatrix}R_x\\ R_y\end{pmatrix}+\begin{pmatrix}\gamma_{13}-\beta_{33}y\\ \gamma_{23}+\beta_{33}x\end{pmatrix}M_z. \tag{23}$$

Hence a condition $u_x(x_{CRES},y_{CRES})=u_y(x_{CRES},y_{CRES})=0$ for $R_x=R_y=0$ and an arbitrary M_z is equivalent to (22) ($\beta_{33}>0$ because of $\hat{\beta}$ is positive definite matrix).

b) It follows from (11), (20), (21) that:

$$\lambda=\mu=0,\ \nu=\gamma_{13}R_x+\gamma_{23}R_y+\beta_{33}M_z \tag{24}$$

(here ν is angle of tooth rotation). If $\mathbf{M}=\mathbf{r}_{CRES}\times\mathbf{R}$, i.e. $M_z=x_{CRES}R_y-y_{CRES}R_x$, then condition $\nu=0$ for arbitrary R_x, R_y is equivalent to (22).

Note.

Let $M_{CRES} = M_z - x_{CRES} R_y + y_{CRES} R_x$ denote a resulting moment of the load about the axis, which is parallel to the axis Z and passes through the CRES. Then from (22) and (23) we obtain:

$$\begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix} = \hat{K}\begin{pmatrix} R_x \\ R_y \end{pmatrix} + \beta_{33}\begin{pmatrix} y_{CRES} - y \\ x - x_{CRES} \end{pmatrix} M_{CRES}, \tag{25}$$

where $\hat{K} = \begin{pmatrix} \alpha_{11} - \gamma_{13}^2/\beta_{33} & \alpha_{12} - \gamma_{13}\gamma_{23}/\beta_{33} \\ \alpha_{12} - \gamma_{13}\gamma_{23}/\beta_{33} & \alpha_{22} - \gamma_{23}^2/\beta_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}^{-1}$.

The elements of matrix $\hat{K}$ expressed in terms of elements of $\hat{A}$ may be derived from (12) and (20). Because of symmetry of matrix $\hat{K}$, it has two mutually perpendicular principal vectors [19]. Directions defined by these vectors are “principal” in the following sense: if applied forces are reduced to resultant with a line of action passing through the CRES and being parallel to one of the “principal” directions, then translation of the tooth occurs *in the direction of the line of action of the resultant*. It should be emphasized that if resultant passes through the CRES but is nonparallel to all the “principal” directions, then translation of the tooth occurs but, generally speaking, in the direction nonparallel to the line of action of the resultant. Let us choose OXYZ so that the point O is coincident with the CRES. Moreover we require directions of axes X and Y to be coincident with the “principal” directions. Then (25) yields:

$$\begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix} = \begin{pmatrix} K_x & 0 \\ 0 & K_y \end{pmatrix}\begin{pmatrix} R_x \\ R_y \end{pmatrix} + \beta_{33}\begin{pmatrix} -y \\ x \end{pmatrix} M_z,$$

where K_x and K_y (eigenvalues of matrix $\hat{K}$) have the physical meanings of measures of periodontal ligament compliances corresponding to the tooth translations in the “principal” directions. Since $\hat{A}$ is positive definite, $\hat{K}$ is also positive definite. Therefore $K_x > 0$, $K_y > 0$ and $K_{11} = \alpha_{11} - \gamma_{13}^2/\beta_{33} > 0$ (the last inequality will be used below).

Center of rotation: the definition, existence, properties

Definition 3.

The center of rotation of a tooth (CRO) is defined as a point rigidly connected with the tooth and satisfying the requirement $\mathbf{u}(\mathbf{r}_{\mathrm{CRO}}) = 0$ under action of a given load.

It follows from the Definition 3 that the position of the CRO in OXYZ depends on applied forces. Further we exclude the trivial case **R**=0, **M**=0 (and thus **u**(**r**)=0 for each tooth point) from consideration.

Theorem 3.

The CRO exists if and only if $\varphi \neq 0$ and $\rho \cdot \varphi = 0$ (ρ, φ are determined from (11)). In this case CRO is nonunique and:

$$\mathbf{r}_{\mathrm{CRO}} = \frac{\varphi \times \rho}{\varphi^2} + \ell\varphi, \tag{26}$$

where ℓ is an arbitrary distance, the set of all the CRO forming rotation axis.

Proof.

a) Let $\varphi = 0$. Since the trivial case $\varphi = 0$, $\rho = 0$ is excluded from consideration, then $\rho \neq 0$ and $\mathbf{u}(\mathbf{r}) = \rho \neq 0$ for an arbitrary **r**. Therefore the CRO does not exist.

b). Let $\mathbf{u}(\mathbf{r}) = \rho + \varphi \times \mathbf{r} = 0$ for some **r**. Taking the scalar product of this relation by φ, we have $\rho \cdot \varphi = 0$. Thus if $\rho \cdot \varphi \neq 0$, then CRO does not exist.

c). Let $\boldsymbol{\varphi}\neq 0$ and $\boldsymbol{\rho}\cdot\boldsymbol{\varphi}=0$. Let us prove that $\mathbf{u}(\mathbf{r})=0$ for each points satisfying (26), and conversely if $\mathbf{u}(\mathbf{r})=0$, then $\mathbf{r}$ is of the form (26). If we assume:

$$\mathbf{r}=\frac{\boldsymbol{\varphi}\times\boldsymbol{\rho}}{\varphi^2}+\ell\boldsymbol{\varphi},$$

then it follows that:

$$\mathbf{u}(\mathbf{r})=\boldsymbol{\rho}+\boldsymbol{\varphi}\times\mathbf{r}=\boldsymbol{\rho}+\frac{\boldsymbol{\varphi}(\boldsymbol{\varphi}\cdot\boldsymbol{\rho})-\boldsymbol{\rho}\varphi^2}{\varphi^2}=0,$$

i.e. $\mathbf{u}(\mathbf{r})=0$ for each point satisfying (26). Further, if $\mathbf{u}(\mathbf{r})=0$ for some $\mathbf{r}$, then $\boldsymbol{\rho}+\boldsymbol{\varphi}\times\mathbf{r}=0$. Taking the vector product of this relation by $\boldsymbol{\varphi}$, we have:

$$\mathbf{r}=\frac{\boldsymbol{\varphi}\times\boldsymbol{\rho}}{\varphi^2}+\frac{\boldsymbol{\varphi}\cdot\mathbf{r}}{\varphi^2}\boldsymbol{\varphi}.$$

Therefore $\mathbf{r}$ is of the form (26).

It is readily shown that the possibility exists of choosing a load so that firstly, CRO exists; secondly, a given point T is CRO; and thirdly, tooth rotates by a given angle about a given axis. In fact, $\boldsymbol{\varphi}$ is prescribed (its direction is along the rotation axis, absolute value is equal to the rotation angle). Let $\rho=\mathbf{r}_T\times\varphi$. Then it follows from (5) that $\mathbf{u}(\mathbf{r}_T)=0$. Given $\boldsymbol{\rho}$ and $\boldsymbol{\varphi}$, one may determine $\mathbf{R}$ and $\mathbf{M}$ using (10).

Consider again the case when the tooth + the periodontal ligament system has the only symmetry plane XY, load belonging to this plane. It follows from (11), (20), (21) that:

$$\zeta=0,\quad \begin{pmatrix}\xi\\ \eta\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{12} & \alpha_{22}\end{pmatrix}\begin{pmatrix}R_x\\ R_y\end{pmatrix}+\begin{pmatrix}\gamma_{13}\\ \gamma_{23}\end{pmatrix}M_z. \tag{27}$$

Then from (24) and (27) it follows that $\boldsymbol{\rho}\cdot\boldsymbol{\varphi}=0$ in this case. A condition $\nu\neq 0$ turns out to be an existence condition of a CRO, rotation axis (26) being perpendicular to the symmetry plane.

<u>Definition 4.</u>

The center of rotation of a tooth in a symmetry plane (CROS) is defined as a point belonging to this plane, the rotation axis (26) passing (if $\nu\neq 0$) through a CROS.

It follows from (24), (26), (27) that the coordinates of CROS may be written as:

$$x_{CROS}=-\frac{\eta}{\nu}=-\frac{\alpha_{12}R_x+\alpha_{22}R_y+\gamma_{23}M_z}{\gamma_{13}R_x+\gamma_{23}R_y+\beta_{33}M_z},$$
$$y_{CROS}=\frac{\xi}{\nu}=\frac{\alpha_{11}R_x+\alpha_{12}R_y+\gamma_{13}M_z}{\gamma_{13}R_x+\gamma_{23}R_y+\beta_{33}M_z}. \tag{28}$$

Note that from (24) and (28) the magnitudes of R_x, R_y, M_z, ensuring prescribed ν, x_{CROS}, y_{CROS} can readily be determined.

As an example of using above obtained formulas let us consider the following problem. The load is applied to the tooth so that $R_y=0$. Determine: the relationship between y_{CROS}-y_{CRES} and M_{CRES}/R_x. Since $M_z=M_{CRES}-y_{CRES}R_x$, then it follows from (22) and (28) that:

$$y_{CROS}-y_{CRES}=\frac{\alpha_{11}-\gamma_{13}^2/\beta_{33}}{\beta_{33}M_{CRES}/R_x}. \tag{29}$$

As it was established discussed above, an inverse proportion coefficient is positive. It should be mentioned that inverse proportion relationship between $y_{CROS} - y_{CRES}$ and M_{CRES} / R_x is in agreement with the studies [2, 10, 11, 14, 15], in which the corresponding graph (hyperbola) is presented.

Conclusion

This study was designed to introduce the exact mathematical definitions of concepts of CRE and CRO (and their modifications CRES, CROS). Using of these concepts may assist an orthodontist in individual examination of patient's teeth and treatment planning. In the future, the authors will apply this theory to developing methods of individual noninvasive location of CRE and CRO as well as to determining forces causing prescribed orthodontic tooth movement.

References

1. BULCKE M.M, DERMAUT L.R., SACHDEVA R.C.L., BURSTONE C.J. The center of resistance of anterior teeth during intrusion using the laser reflection tecnique and holographic interferometry. **Am J Orthod Dentofac Orthop,** 90: 211-220, 1986.
1. 2. BURSTONE C.J., PRYPUTNIEWICZ R.J. Holographic determination of centers of rotation produced by orthodontic forces. **Am J Orthod,** 77: 396-409, 1980.
2. 3. CHRISTIANSEN R.L., BURSTONE C.J. Centers of rotation within the periodontal space. **Am J Orthod,** 55: 353-369, 1969.
3. 4. DEMANGE C. Equilibrium situations in bend force systems. **Am J Orthod Dentofac Orthop,** 98: 333-339, 1990.
4. 5. DERMAUT L.R., KLEUTGHEN J.P.J., De CLERCK H.J.J. Experimental determination of the center of resistance of the upper first molar in a macerated, dry human skull submitted to horizontal headgear traction. **Am J Orthod Dentofac Orthop,** 90: 29-36, 1986.
5. 6. HASKELL B.S., SPENCER W.A., DAY M. Auxiliary springs in continuous arch treatment: Part 1. An analytical study employing the finite-element method. **Am J Orthod Dentofac Orthop,** 98: 387-397, 1990.
6. 7. HOCEVAR R.A. Understanding, planning, and managing tooth movement: orthodontic force system theory. **Am J Orthod,** 80: 457-477, 1981.
7. 8. HURD J.J., NIKOLAI R.J. Centers of rotation for combined vertical and transverse tooth movements. **Am J Orthod,** 70: 551-558, 1976.
8. 9. KUSY R.P., TULLOCH J.F.C. Analysis of moment/force ratios in the mechanics of tooth movement. **Am J Orthod Dentofac Orthop,** 90: 127-131, 1986.
9. 10. NYASHIN M.Y., PECHENOV V.S., RAMMERSTORFER F.G. Determination of optimal orthodontic forces. **Russian J Biomech,** 1: 84–96, 1997.
10. 11. PEDERSEN E., ANDERSEN K., GJESSING P.E. Electronic determination of centres of rotation produced by orthodontic force systems. **European J Orthod,** 12: 272-280, 1990.
11. 12. SCHWARZ A.D. **Biomechanics and occlusion**. Medicine (Moscow), 1994 (in Russian).
12. 13. SOSNIN G.P. Clasp prostheses. Nauka i Tehnika (Minsk), 1981 (in Russian).
13. 14. SMITH R.J., BURSTONE C.J. Mechanics of tooth movement. **Am J Orthod,** 85: 294-307, 1984.
14. 15. TANNE K., KOENIG H.A., BURSTONE C.J. Moment to force ratios and the center of rotation. **Am J Orthod Dentofac Orthop,** 94: 426-431, 1988.
15. 16. VALET R., PAGIN J. The center of resistance. **Rev Orthop DentoFaciale,** 21: 115-129, 1987.
16. 17. McGUINNESS N.J.P., WILSON A.N., JONES M.L., MIDDLETON J. A stress analysis of the periodontal ligament under various orthodontic loadings. **European J Orthod,** 13: 231-242, 1991.
17. 18. PARS L.A. A treatise on analytical dynamics. Heinemann, London, 1964.
18. 19. KORN G.A., KORN T.M. Mathematical handbook. McGraw-Hill Book Company, New York, San Francisco, Toronto, London, Sydney, 1968.

Центр сопротивления и центр вращения зуба: определения, условия существования, свойства

М.А. Осипенко, М.Ю. Няшин, Ю.И. Няшин
(Пермь, Россия)

В настоящей работе введены точные математические определения понятий “центр сопротивления” и “центр вращения зуба”. Эти понятия широко используются в стоматологической литературе, однако определяются нечетко. Показано, что центр сопротивления и центр вращения существуют не всегда. Приведены условия их существования и формулы для определения их положений. Библ. 19.

Ключевые слова: биомеханика зубов, ортодонтия, аналитическая механика, центр сопротивления, центр вращения

Received 10 August 1998