A PROBLEM WITH THE UNILATERAL CONSTRAINTS IN THE THEORY OF BENDING OF FOOT PROSTHESIS

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Abstract: The purpose of this study is to investigate the characteristics of the contact area in the packet of plates under weak bending. The packet is fixed at one end, the lengths of the plates are different. This structure models the framework of foot prosthesis. The assumption that the contact of two adjacent plates takes place only at their ends is verified. This assumption was used in the earlier authors' studies. The plates which are flat in their natural state are considered. The contact area in the system of two plates under arbitrary loading is investigated in full. The load density in this area is found and the conditions for which the area becomes the point are formulated. For the system of arbitrary number of plates the conditions of point contact are formulated for the special case in which the load represents the concentrated force. It is found that the equal strength structure proposed in the earlier authors' study, satisfies these conditions.

Key words: foot prosthesis, packet of plates, weak bending, unilateral constraints, contact area

Introduction

The framework of the foot prosthesis manufactured at the Urals Scientific Research Institute of Composite Materials is shown in Fig. 1. Individual fitting of the prosthesis includes the selection of its size (according to the size of a footwear) and stiffness (as the patient likes). The size is determined by the length of the lowermost plate. Stiffness can be determined, for example, by the deflection of the end of the framework under standard loading. Given the size and the stiffness, we should maximize the durability of the prosthesis by choosing the lengths of the plates (beginning from the second) and their thickness so as to minimize the stresses (for example, the maximum principal stress) in the structure under standard loading.

Finding the shapes of the plates under bending represents the main part of this optimization problem. Given the shapes, one can find the stresses and minimize them. Consider the framework schematically as the packet of thin plates fixed at one end (Fig. 2). The plates are pressed close up to each other (without loading). There is no friction between the plates. All of the plates have the same width (in the direction perpendicular to the picture plane). The given load is applied to the lowermost plate. The model of the linear bending of the packet of plates under the action of the concentrated force applied to the end of the lowermost plate perpendicular to it was proposed in [1,2]. It was assumed that the contact of two adjacent plates under bending took place only at one point (except for the fixed point) at the end of the shorter plate.

But this "hypothesis of the point contact", strictly speaking, cannot be used as an independent assumption. The characteristics of the contact area should be derived from the equations describing the bending of plates. The present study deals with this problem in some special cases.

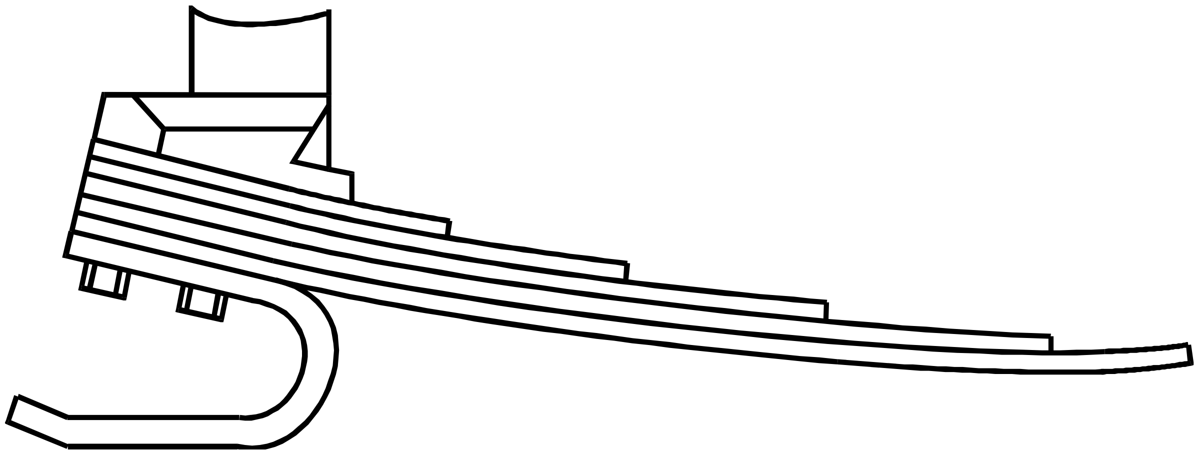

Fig. 1. The framework of the foot prosthesis.

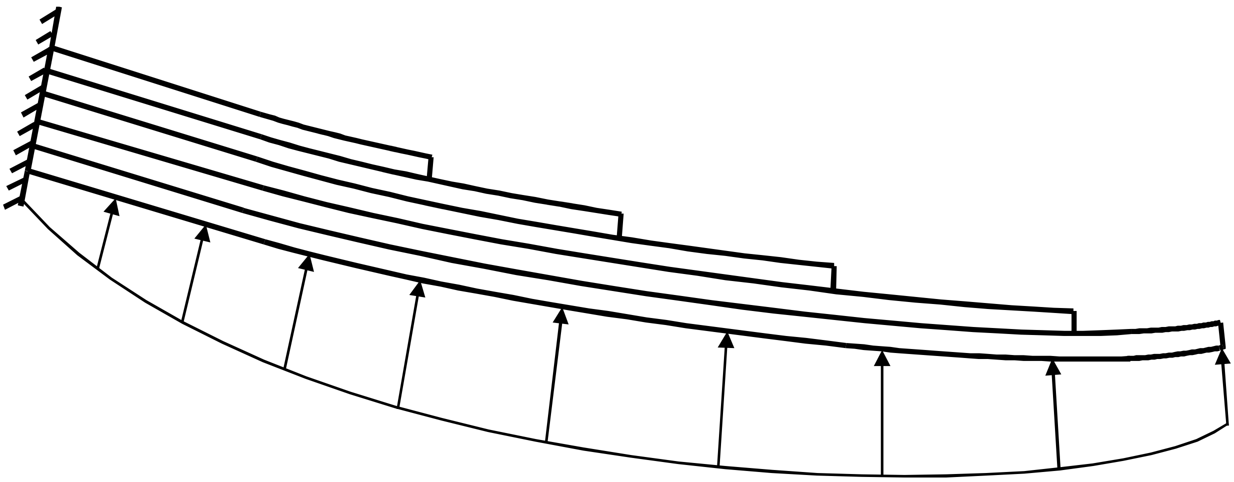

Fig. 2. The simplified scheme of the framework.

Formulation of the problem

Consider the weak bending of two thin plates, which are flat in their natural state (Fig. 3). The load with the given density $q(x)$ is applied to the lower plate perpendicular to it. The shapes of the plates are described by the functions $y_1(x)\ (0 \le x \le L),\ y_2(x)\ (0 \le x \le \ell)$, where $L \ge \ell > 0$ are considered to be equal to the lengths of the plates (according to the weak bending approximation). We assume that the functions $y_1(x),\ y_2(x)$ have continuous second derivatives and that the load density $q(x)$ can be represented in the form of

$$u(x) + \sum_p U_p \delta(x - x_p), \tag{1}$$

where $u(x) \ge 0$ is the finite, piecewise continuous function without pointwise peaks, all the values of $0 < x_p \le L$ are different, $U_p \ge 0$.

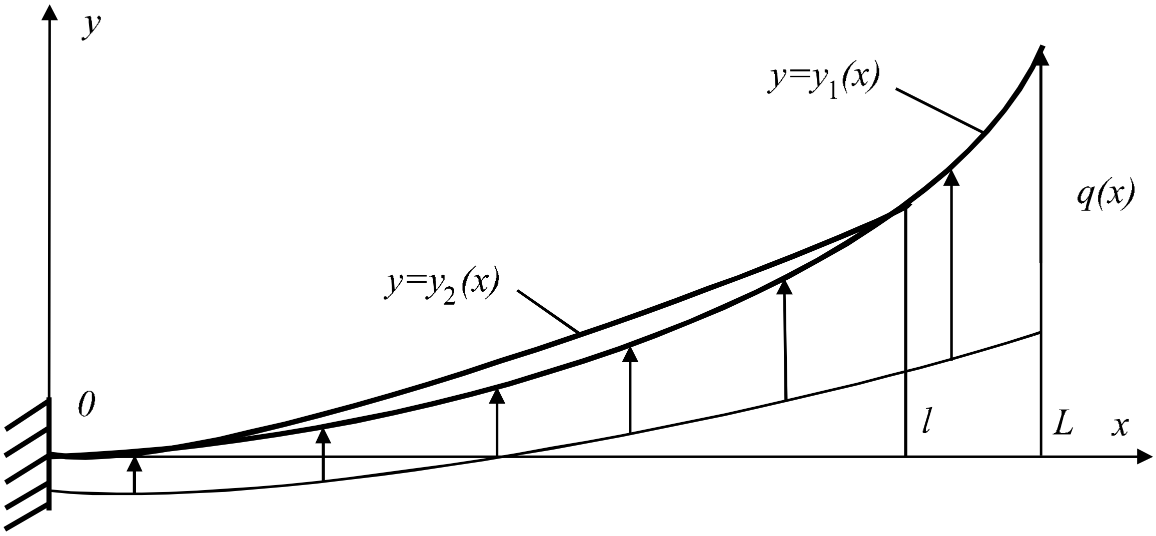

Fig. 3. The scheme of bending of two plates.

It is required to find $y_1(x), y_2(x)$ (without any assumptions about the contact area). According to the least potential energy principle [3], this problem can be reduced to the minimization of the functional

$$\mathbf{E}[y_1, y_2] = \frac{1}{2c_1}\int_0^L (y_1''(\xi))^2 d\xi + \frac{1}{2c_2}\int_0^\ell (y_2''(\xi))^2 d\xi - \int_0^L q(\xi) y_1(\xi) d\xi$$

on the specified set of functions under the conditions

$$y_1(0) = y_2(0) = y_1'(0) = y_2'(0) = 0,$$

$$y_2(x) \ge y_1(x) \text{ for } 0 \le x \le \ell \text{ (unilateral constraints)},$$

where $c_k = 12/Edh_k^3$; E is the elastic modulus; d is the width of the plates; h_k are their thicknesses; $k = 1,\ 2$. Functional $\mathbf{E}$ being strictly convex and the set on which $\mathbf{E}$ is minimized being convex, the solution of the problem formulated is unique. In order to construct this solution, it is convenient to reformulate the problem so that to regard the density $f(x)$ of the forces of interaction between the plates as the function to be found. We assume that this function is of type (1). The functions $y_1(x),\ y_2(x)$ are expressed in terms of $f(x)$ as follows [3]:

$$\begin{cases} y_1(x) = c_1\int_0^L G(x,\xi) q(\xi) d\xi - c_1 \int_0^\ell G(x,\xi) f(\xi) d\xi, \\ y_2(x) = c_2 \int_0^\ell G(x,\xi) f(\xi) d\xi, \end{cases} \quad (2)$$

$$\text{where } G(x,\xi) = \begin{cases} \xi^2 (x - \xi/3)/2 & \text{if } \xi \le x, \\ x^2(\xi - x/3)/2 & \text{if } \xi \ge x. \end{cases} \quad (3)$$

It is evident that $G \ge 0$. Introduce the notation $\rho(x) = y_2(x) - y_1(x)$ and note that $\rho(x) \ge 0$, $f(x) \ge 0$ and $\rho(x) = 0$ if $f(x) > 0$. Then, using (2), we come to the following problem.

Problem.

It is required to find $f(x)$ $(0 \le x \le \ell)$, which is of type (1) and should satisfy the conditions

$$\rho(x) = (c_1 + c_2)\int_0^\ell G(x,\xi) f(\xi) d\xi - c_1 \int_0^L G(x,\xi) q(\xi) d\xi \begin{cases} = 0 & \text{if } f(x) > 0, \\ \ge 0 & \text{if } f(x) = 0. \end{cases} \quad (4)$$

Solution of the problem

We will introduce the notation

$$J = \int_{\ell}^{L} (\xi - \ell) q(\xi) d\xi \geq 0, \tag{5}$$

$$\Phi(\Lambda) = J - \frac{1}{(\ell - \Lambda)^2} \int_{\Lambda}^{\ell} (\xi - \Lambda)(\ell - \xi)(2\ell - \Lambda - \xi) q(\xi) d\xi , \tag{6}$$

where $0 \leq \Lambda < \ell$.

Theorem 1.

(i) If $J = 0$, then the solution of the problem is

$$f(x) = \frac{c_1}{c_1 + c_2} q(x) . \tag{7}$$

(ii) If $J > 0$ and $\Phi(0) \geq 0$, then the solution of the problem is

$$f(x) = F\delta(x - \ell) , \tag{8}$$

$$\text{where } F = \frac{c_1}{c_1 + c_2} \frac{1}{G(\ell, \ell)} \int_0^L G(\ell, \xi) q(\xi) d\xi . \tag{9}$$

(iii) If $J > 0$ and $\Phi(0) < 0$, then the solution of the problem is

$$f(x) = F\delta(x - \ell) + P\delta(x - \lambda) + \begin{cases} \dfrac{c_1}{c_1 + c_2} q(x) & \text{if } \ 0 \leq x < \lambda , \\ 0 & \text{if } \ x \geq \lambda , \end{cases} \tag{10}$$

where $0 < \lambda < \ell$ is the root of the equation $\Phi(\Lambda) = 0$, (11)

$$F = \frac{c_1}{c_1 + c_2} \frac{1}{\ell - \lambda} \int_{\lambda}^{L} (\xi - \lambda) q(\xi) d\xi , \quad P = \frac{c_1}{c_1 + c_2} \frac{1}{\ell - \lambda} \int_{\lambda}^{L} (\ell - \xi) q(\xi) d\xi . \tag{12}$$

Proof.

(i) It is evident that $f(x)$ is of type (1). Substituting (7) into (4) yields

$$\rho(x) = -c_1 \int_{\ell+0}^{L} G(x, \xi) q(\xi) d\xi .$$

The notation $\ell + 0$ means that the segment of integrating does not include the point $\xi = \ell$. This indication is essential because $q(x)$ may contain δ-functions. Because $q(x)$ is of type (1), it follows from (5) that at $J = 0$ $q(x) \equiv 0$ for $\ell < x \leq L$. Therefore the function $\rho(x) \equiv 0$ and the conditions (4) are satisfied.

(ii) It is evident that $F \geq 0$, whence it follows that $f(x)$ is of type (1). Substituting (8), (9) into (4) yields

$$\rho(x) = \frac{c_1}{G(\ell, \ell)} \int_0^L R(\ell, x, \xi) q(\xi) d\xi , \tag{13}$$

$$\text{where } \quad R(\ell, x, \xi) = G(x, \ell) G(\ell, \xi) - G(\ell, \ell) G(x, \xi) . \tag{14}$$

It follows from (13), (14) that $\rho(\ell) = 0$. Since $f(x) > 0$ only at $x = \ell$, we should establish that

$$\rho(x) \geq 0 \text{ for } 0 \leq x \leq \ell$$

in order for (4) to be proved. It may be derived from (3), (14) that

$$R(\ell, x, \xi) = \frac{1}{12} \ell^2 x^2 (\ell - x)(\xi - \ell) \quad \text{for } \ 0 \leq x \leq \ell \leq \xi \leq L . \tag{15}$$

Since $\Phi(0) \geq 0$, it follows from (5), (6), (15) that

$$\int_{\ell}^{L} R(\ell, x, \xi) q(\xi) d\xi \geq \int_0^{\ell} \tilde{R}(\ell, x, \xi) q(\xi) d\xi , \tag{16}$$

where $\tilde{R}(\ell,x,\xi)=\dfrac{1}{12}x^2(\ell-x)\xi(\ell-\xi)(2\ell-\xi)$.

Then (13), (16) yields

$$\rho(x)\geq\frac{c_1}{G(\ell,\ell)}\int_0^{\ell}\left(R(\ell,x,\xi)+\tilde{R}(\ell,x,\xi)\right)q(\xi)d\xi .$$

Taking (3) into account, one can find the explicit expression for $R+\tilde{R}$ (different expressions for $0\leq\xi\leq x$ and for $x\leq\xi\leq\ell$) and prove that $R+\tilde{R}\geq 0$. Hence $\rho(x)\geq 0$ for $0\leq x\leq\ell$.

Note that the condition $J>0$ has not been used in the proof of (ii), hence (8), (9) are valid also at $J=0$. One can easily find that the condition $J=0$ together with the condition $\Phi(0)\geq 0$ correspond to the case $q(x)=F_0\delta(x-\ell)$ in which the solutions (7) and (8) coincide.

(iii) Prove first of all that the value λ is defined correctly, i.e. the existence and uniqueness of the root of the equation (11). Because $q(x)$ is of type (1), it follows from (6) that $\Phi(\Lambda)$ is the continuous function, $\Phi'(\Lambda)$ is the piecewise continuous function and

$$\Phi(0)<0, \tag{17}$$

$$\lim_{\Lambda\to\ell}\Phi(\Lambda)=J>0, \tag{18}$$

$$\Phi'(\Lambda)=\frac{2}{(\ell-\Lambda)^3}\int_{\Lambda}^{\ell}(\ell-\xi)^3q(\xi)d\xi\geq 0 \tag{19}$$

for $0\leq\Lambda<\ell$ (at the points, where $\Phi'(\Lambda)$ exists). Then the existence of the root $0<\lambda<\ell$ of the equation (11) follows from (17)-(19). But the inequality (19) is not strict and the uniqueness of the root should be proved separately. It follows from (19) that if $\Phi'(\Lambda_*)=0$ at certain $0\leq\Lambda_*<\ell$, then $\Phi'(\Lambda)\equiv 0$ (and $\Phi(\Lambda)\equiv const$) for $\Lambda_*\leq\Lambda<\ell$. However according to (18) this constant cannot be equal to zero. Hence the root λ is unique.

It is evident that $F\geq 0$. Prove that $P\geq 0$. We rewrite (12) in the form

$$P=\frac{c_1}{c_1+c_2}\frac{1}{\ell-\lambda}\left(\int_{\lambda}^{\ell}(\ell-\xi)q(\xi)d\xi-\int_{\ell}^{L}(\xi-\ell)q(\xi)d\xi\right).$$

We treat the second integral using (5), (6), (11) and, taking (19) into account, we find

$$P=\frac{1}{2}\frac{c_1}{c_1+c_2}\Phi'(\lambda-0)\geq 0 .$$

Hence $f(x)$ is of type (1). Substituting (10), (12) into (4) yields

$$\rho(x)=\frac{c_1}{\ell-\lambda}\int_{\lambda}^{L}Q(x,\xi,\ell,\lambda)q(\xi)d\xi , \tag{20}$$

where $Q(x,\xi,\ell,\lambda)=(\ell-\xi)G(x,\lambda)+(\xi-\lambda)G(x,\ell)-(\ell-\lambda)G(x,\xi)$. (21)

According to (10), $f(x)$ may be positive only for $0\leq x\leq\lambda$ and $x=\ell$. For these x values $\rho(x)=0$. Indeed, using (20), (21), (3), (6), one can find that $Q(x,\xi,\ell,\lambda)=0$ for $x\leq\lambda<\ell,\ \lambda\leq\xi$ and

$$\rho(\ell)=-\left(c_1/6\right)(\ell-\lambda)^2\Phi(\lambda)=0 .$$

Let now x in (20) be in the limits of $\lambda<x\leq\ell$. Consider the function $\tilde{\rho}(x)=\rho(x)/(x-\lambda)^3$. Because $q(x)$ is of type (1), one can find from (20), (21), (3) that for the specified x $\tilde{\rho}(x)$ has continuous derivative and

$$\tilde{\rho}(\ell)=0,\quad \tilde{\rho}'(x)=-\frac{c_1}{2(x-\lambda)^4}\int_{\lambda}^{x}(x-\xi)^2(\xi-\lambda)q(\xi)d\xi\leq 0 . \tag{22}$$

It follows from (22) that $\tilde{\rho}(x)\geq 0$ and, consequently, $\rho(x)\geq 0$ for $\lambda<x\leq\ell$. Hence (4) is satisfied.

Note.

It follows from the theorem proved that in the case of two plates the hypothesis of the point contact is valid at the condition $\Phi(0)\geq 0$. If $q(x)=F_0\delta(x-L)$ then this condition is satisfied.

Some generalizations of the obtained solution

Consider the weak bending of $N \geq 2$ plates, which are flat in their natural state. The lengths of the plates are $\ell_k > 0$, $1 \leq k \leq N$ (numeration begins at the lowermost plate). The sequence ℓ_k is non-increasing. Let the load density be

$$q(x) = F_0 \delta(x - \ell_1) . \tag{23}$$

Theorem 2.

The sufficient conditions for the validity of the hypothesis of the point contact in the case under consideration are

$$1 - \frac{\gamma_{k+1}}{1-} \frac{\gamma_{k+2}}{1-} \ldots - \frac{\gamma_{N-2}}{1 - \gamma_{N-1}} \geq \frac{\beta_k (1 - 2\alpha_{k+1}\beta_{k+1})}{\beta_k - \alpha_{k+1}\beta_{k+1}} , \tag{24}$$

where $1 \leq k \leq N-1$,

$$\alpha_k = \frac{\ell_{k+1}}{\ell_k} , \ \beta_k = \frac{3 - \alpha_k}{4} , \ \gamma_k = \frac{4c_k^2 \alpha_k \beta_k^2}{(c_{k-1} + c_k)(c_k + c_{k+1})} . \tag{25}$$

For $k = N-1$ the catenary fraction in the left hand side of (24) should be set to 1; ℓ_{N+1} should be set to zero.

Proof.

For N plates we have $(N-1)$ functions $f_k(x)$ (the density of the forces of interaction between k-th and $(k+1)$-th plates) to be found, which should be of type (1) and should satisfy the conditions analogous to (4)

$$\rho_k(x) = -c_k \int_0^{\ell_k} G(x,\xi) f_{k-1}(\xi) d\xi + (c_k + c_{k+1}) \int_0^{\ell_{k+1}} G(x,\xi) f_k(\xi) d\xi - \\ - c_{k+1} \int_0^{\ell_{k+2}} G(x,\xi) f_{k+1}(\xi) d\xi \begin{cases} = 0 & \text{if} \quad f_k(x) > 0, \\ \geq 0 & \text{if} \quad f_k(x) = 0, \end{cases} \tag{26}$$

where $1 \leq k \leq N-1$, $0 \leq x \leq \ell_{k+1}$. It should be considered that $f_0(x) = q(x)$, $f_{N+1}(x) = 0$. Prove that

$$f_k(x) = F_k \delta(x - \ell_{k+1}) \tag{27}$$

(i.e. the hypothesis of the point contact is valid) for the conditions (23), (24), where F_k are found from the equations

$$\rho_k(\ell_{k+1}) = 0 , \ 1 \leq k \leq N-1 . \tag{28}$$

Substituting (27) into (26) yields

$$\rho_k(x) = -c_k G(x, \ell_k) F_{k-1} + (c_k + c_{k+1}) G(x, \ell_{k+1}) F_k - c_{k+1} G(x, \ell_{k+2}) F_{k+1} . \tag{29}$$

Put $x = \ell_{k+1}$ in (29) and introduce φ_k according to the formula

$$F_k = F_0 \varphi_k \prod_{p=1}^{k} \frac{2 c_p \beta_p}{\alpha_p (c_p + c_{p+1})} . \tag{30}$$

Then (28) may be reduced to the system of linear algebraic equations in the unknown φ_k

$$-\varphi_{k-1} + \varphi_k - \gamma_{k+1} \varphi_{k+1} = 0 , \ 1 \leq k \leq N-1 , \tag{31}$$

where $\varphi_0 = 1$, $\varphi_N = 0$. Using (25), one can find that the principal minors of matrix of system (31) are positive (matrix should be symmetrized by means of linear transformation, then the corresponding quadratic form should be constructed and it proves positive definite). Hence the solution of (31) is unique and

$$\varphi_k > 0\,, \tag{32}$$

$$\frac{\varphi_{k-1}}{\varphi_k} = 1 - \frac{\gamma_{k+1}}{1-}\frac{\gamma_{k+2}}{1-}\ldots\frac{\gamma_{N-2}}{-1-\gamma_{N-1}}\,, \tag{33}$$

where $1 \le k \le N-1$ and for $k = N-1$ the catenary fraction in the right hand side should be set to 1. It follows from (32), (30), (27) that $f_k(x)$ is of type (1).

Further, $f_k(x) > 0$ only at $x = \ell_{k+1}$ and $\rho_k(\ell_{k+1}) = 0$. Therefore we should establish that

$$\rho_k(x) \ge 0 \text{ for } 0 \le x \le \ell_{k+1} \tag{34}$$

in order for (4) to be proved.

Taking into account that (29) (according to (3)) leads to the different explicit forms of $\rho(x)$ for $0 \le x \le \ell_{k+2}$ and for $\ell_{k+2} \le x \le \ell_{k+1}$, one can find from (29), (3) two conditions equivalent to (34)

$$x^2\overline{\rho}_k(x) \ge 0 \text{ for } 0 \le x \le \ell_{k+2}\,, \tag{35}$$

$$\rho_k(x) \ge 0 \text{ for } \ell_{k+2} \le x \le \ell_{k+1}\,, \tag{36}$$

$\overline{\rho}_k(x)$ being the linear function. Hence (35) is equivalent to

$$\overline{\rho}_k(0) \ge 0\,, \tag{37}$$

$$\overline{\rho}_k(\ell_{k+2}) \ge 0\,. \tag{38}$$

Using the explicit form of $\rho_k(x)$ for $\ell_{k+2} \le x \le \ell_{k+1}$ and carrying out an easy analysis, one can find that (36) follows from (38). Hence (34) follows from (37), (38). Using the explicit form of $\overline{\rho}_k(x)$ and also (31), (25) one can reduce (37) to the inequality

$$\frac{\varphi_{k-1}}{\varphi_k} \ge \frac{\beta_k(1-2\alpha_{k+1}\beta_{k+1})}{\beta_k - \alpha_{k+1}\beta_{k+1}}\,. \tag{39}$$

Inequality (39) follows from (33), (24). Hence (37) is proved. The analogous treatment on (38) using (31), (25) leads to a conclusion that (38) follows from (39). Therefore (34) and, consequently, (4) are proved.

Consequence.

The equal strength structure of the framework (corresponding to the concentrated force as a load) was proposed in [1]. The structure consisted of *N* plates, which were flat in their natural state. In such structure, the maximum principal stress is the same for all of the plates and it is constant along the plate from the fixed point up to the point of contact with the upper plate. Then it diminishes down to zero at the free end of the plate. The hypothesis of the point contact was used to calculate the parameters of the equal strength structure. Let us prove that this hypothesis is valid in this case.

The parameters of the structure are defined as follows [1]

$$\ell_k = \ell_1 \prod_{p=1}^{k-1} x_p\,,\ h_k = A\big(\ell_k(1-x_k)\big)^{1/2}\,, \tag{40}$$

where $A = \left(\dfrac{6F_0}{Ed\Delta}\right)^{1/3} \ell_1^{1/2}\left(\dfrac{1}{x_0}-1\right)^{-1/6}$; Δ is the deflection of the end of the lowermost plate under the action of the concentrated force F_0; Δ, F_0, ℓ_1 are given ; the numbers x_p are found recurrently

$$x_N = 0,\ \ x_p = \left(1 + (1-x_{p+1})\left(1-(1-x_{p+1})^2/3\right)^{-2}\right)^{-1}. \tag{41}$$

One can find from (40), (41), (25), (31) that

$$\frac{\varphi_{k-1}}{\varphi_k} = \frac{2c_k\beta_k}{\alpha_k(c_k + c_{k+1})}\,. \tag{42}$$

It follows from (42), (33), (24) that the hypothesis of the point contact is valid in the case under consideration if

$$\frac{2c_k}{\alpha_k (c_k + c_{k+1})} \geq \frac{1 - 2\alpha_{k+1}\beta_{k+1}}{\beta_k - \alpha_{k+1}\beta_{k+1}} \ , \ 1 \leq k \leq N-1 . \qquad (43)$$

Substituting (40) into (25) and then into (43) and taking (41) into account one can find that (43) is equivalent to the inequalities $x_k (3 - x_k) \geq 0 \ \ (1 \leq k \leq N-1)$, which can be easily derived from (41).

Conclusion

The approach to studying the characteristics of the contact area in the packet of plates under joint bending was proposed. This approach allowed one to investigate the contact area in full in the system of two plates which were flat in their natural state. The conditions of the point contact were found for the arbitrary number of plates. It was proved that these conditions were satisfied for equal strength structure of the foot prosthesis framework, as it had been hypothesized earlier. A first step was made towards providing a full description of the characteristics of the contact area in the packet of arbitrary number of plates.

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Задача с односторонними ограничениями в теории изгиба протеза стопы

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Целью настоящей работы является исследование характера контакта пластин при слабом изгибе их пачки, защемленной на одном конце; длины пластин различны. Такая конструкция моделирует каркас протеза стопы. Проверяется выполнение принятого в предыдущих работах авторов предположения, что контакт происходит только в точках, расположенных на концах пластин. Рассматриваются пластины, плоские в естественном состоянии. Полностью изучен контакт в системе двух пластин для произвольной нагрузки. Показано, что может существовать область контакта и найдена плотность сил взаимодействия пластин в этой области, а также получены условия, при которых область вырождается в точку. Для системы произвольного числа пластин найдены условия точечного контакта в случае, когда нагрузка является сосредоточенной силой. Установлено, что эти условия выполняются для равнопрочной конструкции, предложенной авторами ранее. Библ. 3.

Ключевые слова: протез стопы, пачка пластин, слабый изгиб, односторонние ограничения, область контакта

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