MIXED-MODE LOADING OF THE CRACKED PLATE UNDER PLANE STRESS CONDITIONS

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ABSTRACT
The paper is devoted to the stress-strain analysis near the crack tip in a power-law material under mixed-mode loading. In the paper by the use of the eigenfunction expansion method the stress-strain state near the crack tip under plane stress conditions is found. The type of the mixed-mode loading is specified by the mixity parameter which is varying from 0 to 1. The value of the mixity parameter corresponding to Mode II crack loading is equal to 0, whereas the value corresponding to Mode I crack loading is equal to 1. It is shown that the eigenfunction expansion method results in the nonlinear eigenvalue problem. The numerical solution of the nonlinear eigenvalue problem for all the values of the mixity parameter and for all practically important values of the strain hardening (or creep) exponent is obtained. It is found that the mixed-mode loading of the cracked plate gives rise to a change of the stress singularity in the vicinity of the crack tip. The mixed-mode loading of the cracked plate results in the new asymptotics of the stress-strain fields which is different from the classical Hutchinson-Rice-Rosengren stress field. The approximate solution of the nonlinear eigenvalue problem is either obtained by the perturbation theory technique (small parameter method). In the framework of the small parameter method the small parameter presenting the difference between the eigenvalue of the nonlinear problem and the unperturbed linear problem is introduced. The analysis carried out shows clearly that the stress singularity in the vicinity of the crack tip is changing under mixed-mode deformation in the case of plane stress conditions. The angular distributions of the stress and strain components (eigenvalue functions) in the full range of values of the mixity parameter are given.

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Introduction on Mixed-Mode Deformation of Structural Elements with Cracks

Determination of the stress-strain state in the vicinity of the crack tip is one of priority problems in fracture mechanics now [1-5]. It is an important issue, from the theoretical [1-3], computational [4] and experimental points of view [5]. Recently the analysis of the stress-strain state at a crack tip under mixed loading at a simultaneous exposure of the normal tension and antiplane shear of the materials with nonlinear constitutive equations is of a particular interest [6-14]. It is obvious that the superposition principle of the solutions, corresponding to the opening mode crack and the antiplane shear crack, is correct only for linear elastic materials. For materials with nonlinear (for example, often used power-law) determining equations, it is necessary to develop new methods and approaches to analyze the stress-strain state. The article presents the numerical and approximate analysis of stress and deformation fields near the crack tip under mixed-mode loading (tensile and shear loadings) on the assumption of the plain stress state implementation. It is worth mentioning that the angle distributions and characteristics of mechanical fields in the vicinity of the crack tip under mixed-mode deformation in the plane strain state are studied quite thoroughly [6, 7, 8]. The work in [6] presents the developed method and calculation results of elastoplastic coefficients of the stress intensity in a full range of mixed deformation forms, starting from the opening fracture mode, up to pure antiplane shear. In [6], the state of an arbitrarily oriented rectilinear crack is considered in the form of a mathematical cut under biaxial loading of different intensities. The numerical solution is based on the use of the equation of strain compatibility, represented by the Airy stress function and its derivatives. The behavior of the elastoplastic material corresponds to the Ramberg-Osgood-model (when the power law of the deformation plasticity theory is used for the description of plastic deformation). Based on the calculations in [6], the nature of influence of the type of the mixed-mode loading and plastic material properties, described with the parameter of strain hardening, is established. In [7], the stress-strain state analysis is made for the area adjacent to the tip of the inclined crack, in the specimens of three geometries: the cross-shaped specimen under biaxial loading, the plate with a central inclined crack and a compact single-edge-notch specimen. Under various combinations of loading conditions and orientation of the initial rectilinear crack, a full range of mixed-mode types of deformation from the opening fracture mode to pure shear is reproduced for a plane problem. On the basis of the presented numerical calculations, a complex influence of the biaxiality of the nominal stresses, the initial orientation angle of the defect, the relative crack length, and the geometry of the studied specimen on the stress fields and parameters of the stress-strain state is established. A comparison is made between the solution obtained and the one-parameter classical Hutchinson-Rice-Rosengren type (HRR) solution, which confirms the necessity of taking into account the higher-order terms in the model representations for the case of mixed types of deformation. In [8], static and low-cycle deformation for various variants of biaxial loading are considered, and methods for determining the direction of the crack growth for mixed-mode biaxial loading are presented. In [9], the results of the experimental studies of the plastic zones’ development and steel damage under shear, detachment, mixed-mode loading and off-centre cyclic loading are given. The effect of the shear component on the evolution of zones of plastic deformation, mechanical and acoustic properties (parameters of acoustic emission, velocity and attenuation of ultrasonic waves) is established. The article [10] is devoted to the problems of determining higher approximations in asymptotic decompositions of stress fields in the vicinity of the inclined crack tip, in which the analytical expressions are proposed for the stress field near the tip of the crack under mixed-mode loading, based on the complete solution of M.Williams, and the application of the proposed expressions for the determination of the stress-strain state in the welded joint. In [11], based on the classical solution of the linear fracture mechanics on the loading of a plate with a crack of finite length, analytical expressions are obtained for all the coefficients of the asymptotic decomposition of the stress field as functions of the applied tensile and shear load and crack length. With the help of the analytical expressions found, stress distributions are constructed for an arbitrary combination of detachment and antiplane shear. Works [12, 13] are devoted to fatigue testing methods for steel cross-shaped specimens with a surface crack under biaxial loading. The authors proposed specimens and facilities for the implementation of the mixed-mode deformation of the specimen in its working part. In these studies it was shown that the proposed experimental investigation technique makes it possible to evaluate the susceptibility of various materials to biaxial loading with the development of fatigue cracks. A study of the characteristics of the fatigue crack growth of structural steel under mixed-mode loading was performed in [14], where the values of T-stresses and K-calibration functions were determined on the basis of the numerical calculations of the stress-strain state in a cross-shaped specimen with a central crack. The experiments proved the influence of the type of the stressed state on the growth rate of the crack under biaxial deformation. In [15], an asymptotic representation of the stress-strain state and the field of continuity in the vicinity of the crack tip in the specimen under mixed-mode deformation conditions for the case of a plane deformed state were obtained. Based on the self-similar representation and the hypothesis of the formation of the fully defragmented area (dispersed) material near the crack tip, the distribution of stresses, strain rates and continuity in the stationary crack in a damaged medium in a full range of mixed forms of deformation (from pure shear to opening fracture mode) is obtained. Higher approximations in asymptotic decompositions of stress fields, strain rates of
creep and continuity are built. However, when determining the stress-strain state near the tip of a crack under plane stressed state, certain difficulties arise associated with solving a nonlinear eigenvalue problem, to which the eigenfunction expansion method leads. The present work is aimed at overcoming these difficulties. It should be noted that the problems of determining the plane stressed state for bodies with notches have not been sufficiently studied and are being actively investigated [16-19].

Mathematical Problem Setting. Main Equations. Mixity Parameter of Loading

For the problem of mixed loading of a plate under plane stress state, the master equations in a polar coordinate system with a pole at the crack tip take the form of the balance equation

\[ r\sigma_{rr} + \sigma_{r0} + \sigma_{r\theta} - \sigma_{00} = 0, \quad r\sigma_{\theta r} + \sigma_{\theta 0} + 2\sigma_{\theta\theta} = 0, \] (1)

the compatibility condition of deformations

\[ 2(r\varepsilon_{0r} + \theta)_{r} = \varepsilon_{r0,0} - r\varepsilon_{r,0} + r(r\varepsilon_{00})_{rr}, \] (2)

The constitutive equations of the power law under the assumption of a plane stress state implementation

\[ \varepsilon_{rr} = B\sigma_{r}^{\prime -1}(2\sigma_{0r} - \sigma_{00})/2, \]
\[ \varepsilon_{r0} = B\sigma_{r}^{\prime -1}(2\sigma_{0\theta} - \sigma_{00})/2, \]
\[ \varepsilon_{\theta\theta} = 3B\sigma_{r}^{\prime -1}\sigma_{\theta\theta}/2, \] (3)

where the intensively of stresses is determined by the formula \( \sigma_{r}^{\prime} = \sigma_{rr} + 2\sigma_{0r} - \sigma_{r0} + 3\sigma_{\theta\theta}, \) \( B,n \) are the material constants. The solution of problem (1)-(3) is to satisfy the classical conditions for the absence of surface forces on the crack edges \( \sigma_{\theta r}(r, \theta = \pm \pi) = 0, \)
\( \sigma_{r0}(r, \theta = \pm \pi) = 0. \) The type of the mixed-mode loading is set by the mixity parameter

\[ M^r = (2/\pi)\arctg\lim_{r\to 0}\sigma_{\theta r}(r, \theta = 0)/\sigma_{r0}(r, \theta = 0). \] (4)

The mixity parameter of loading takes the value, equal to one, for the opening mode crack; the value equal to zero, for the antiplane shear crack; for all the intermediate types of loading \( 0 < M^r < 1. \) In polar coordinates, the components of the stress tensor are expressed through the Airy stress function: \( \chi(r, \theta) = \sigma_{00} = \chi_{rr}, \quad \sigma_{r} = = \chi_{r}, \quad \sigma_{\theta} = -\chi_{\theta}/r^2, \quad \sigma_{\theta\theta} = -(-\chi_{\theta})_{r}. \) The asymptotic decomposition of the the Airy stress function \( \chi(r, \theta) \) in the vicinity of the crack tip \( r \to 0 \) is found in the following way

\[ \chi(r, \theta) = Kr^{l-1}f(0), \] (5)

where \( K \) is the amplitude factor (scale factor), depending on the geometry of the specimen with a crack and the system of applied loadings.

Due to the asymptotic representation (5), the components of the stress tensor in the vicinity of the crack tip are written as \( \sigma_{r}(r, \theta) = Kr^{l-1}\hat{\sigma}_{r}(0) \) or

\[ \sigma_{rr}(r, \theta) = Kr^{l-1}[(\lambda + 1)f(0) + f^*(0)], \]
\[ \sigma_{r\theta}(r, \theta) = Kr^{l-1}(\lambda + 1)f(0) \],
\[ \sigma_{\theta\theta}(r, \theta) = -Kr^{l-1}f^*(0). \] (6)

The components of strain tensor in the vicinity of the crack tip when \( r \to 0, \) according to (3) are written as

\[ \varepsilon_{r}(r, \theta) = \frac{1}{2}BK_{n}r^{l-1}\varepsilon_{0}(0) \] or in the explicit form

\[ \varepsilon_{rr}(r, \theta) = \frac{1}{2}BK_{n}r^{l-1}\varepsilon_{0}(0) \]
\[ \varepsilon_{r\theta}(r, \theta) = \frac{3}{2}BK_{n}r^{l-1}\varepsilon_{0}(0) \]

and the compatibility condition for deformations (2) leads to the fourth-order nonlinear ordinary differential equation with respect to function \( f(0) : \)

\[ f^N f^2 \left\{ \begin{array}{l}
(n-1)[(\lambda + 1)(2-\lambda)f + 2f^*]^2 / 2 + 2f^2 + \\
+ 6[(\lambda - 1)n + 1]f^2 h^* f' + f^2 f'^* + \\
+ (n-1)(n-3) h^2 [(\lambda + 1)(2-\lambda)f + 2f^*]^2 + \\
+ [(\lambda + 1)f + f^*][\lambda + 1)f + f^*] ^2 + \\
+ [(\lambda + 1)f + f^*][\lambda + 1)f + f^*] ^2 - \\
- (\lambda + 1)^2 f^2 f^* / 2 - [(\lambda + 1)f + f^*] ^2 (\lambda + 1)f f' - \\
- \frac{1}{2} [(\lambda + 1)f + f^*][\lambda + 1)f + f^*] ^2 + \\
+ 2(n-1)f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] + \\
+ f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] + \\
- (\lambda - 1)n f^2 [(\lambda + 1)(2-\lambda)f + 2f^*] + \\
+ [(\lambda - 1)n + 1]f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] = 0, \end{array} \right. \] (7)

\[ -\frac{1}{2} [(\lambda + 1)f + f^*][\lambda + 1)f + f^*] ^2 + \\
+ 2(n-1)f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] + \\
+ f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] + \\
+ f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] + \\
+ [(\lambda - 1)n + 1]f^2 h^* [(\lambda + 1)(2-\lambda)f + 2f^*] = 0, \] where the abbreviate notations are used

\[ f_e = \sqrt{[(\lambda + 1)f + f^*]^2 + (\lambda + 1)^2 f^2} - \\
- [(\lambda + 1)f + f^*] [(\lambda + 1)f + 3\lambda^2 f^*]. \]
The boundary conditions applied to function \( f(\theta) \), come from the conditions of the absence of traction at the crack edges:

\[
f(\theta = \pm \pi) = 0, \quad f'(\theta = \pm \pi) = 0.
\]  

Thus, the eigenfunction expansion method (5) leads to a nonlinear eigenvalue problem: it is necessary to find the values of \( \lambda \), at which there are non-trivial solutions of the nonlinear ordinary differential equation (7), subjected to the boundary conditions (8). When constructing solutions for the opening fracture mode and antiplane shear cracks, the symmetry (for a type I crack) or antisymmetry (for a type II crack) is used for solutions with respect to the ray \( \theta = 0 \) and the solution is constructed for only one of the half-planes, for example, for the upper one, when \( 0 \leq \theta \leq \pi \). When analyzing the stress-strain state at the crack tip under mixed-mode deformation, the superposition principle of solutions is obviously not valid, and it is necessary to look for a solution in the entire plane \(-\pi \leq \theta \leq \pi\), so that the considerations of symmetry or antisymmetry cannot be used. To construct a numerical solution, it is necessary to take into account the value of the mixed load parameter, which defines the type of the mixed loading. For this purpose, within the present approach, we propose to construct a numerical solution of equation (7) on the interval \([0, \pi]\) with boundary conditions

\[
f(\theta = 0) = 1, \quad f'(\theta = 0) = (\lambda + 1)/\tan(M^p\pi/2),
\]

\[
f(\theta = \pi) = 0, \quad f'(\theta = \pi) = 0.
\]  

The first boundary condition is the normalization requirement, which can be imposed by virtue of the homogeneity of equation (7); the second condition follows from relation (4), which sets the value of the mixed-mode loading parameter. At the next stage the numerical solution of the nonlinear differential equation (7) is built on the interval \([-\pi, 0]\) with boundary conditions

\[
f(\theta = -\pi) = 0, \quad f'(\theta = -\pi) = 0, \quad f(\theta = 0) = 1,
\]

\[
f'(\theta = 0) = (\lambda + 1)/\tan(M^p\pi/2).
\]  

A similar approach was implemented in [15] to determine stress fields in the vicinity of the crack tip under mixed loading in the plane strain state. As a rule, during the construction of a numerical solution, it is assumed that the eigenvalue of the non-linear eigenvalue problem under consideration is equal to the eigenvalue of the HRR problem \( \lambda = n/(n+1) \). However, during the construction of a numerical solution for equation (7), it turned out that the radial stress \( \sigma_{r1}(r, \theta) \) at undergoes a break, whereas for the opening fracture modes and antiplane shear, i.e. for \( M'' = 1 \) and \( M' = 0 \), the stress field is continuous. The calculations performed earlier for cracks in conditions of mixed-mode deformation for the case of the plane deformed state [6-8] lead to continuous distributions of radial stress \( \sigma_{rr}(r, \theta) \) at \( \theta = 0 \). To obtain the asymptotic solution to the nonlinear eigenvalue problem and to analyze the behavior of the radial stress, let us turn to the small parameter method, which is often used in solving eigenvalue problems [20-23].

**Perturbation Technique. Solution of the Nonlinear Eigenvalue Problem**

One of the effective methods for solving eigenvalue problems is the asymptotic theory and perturbation methods [20-23]. Let us introduce a small parameter \( \varepsilon = \lambda - \lambda_0 \), which is the difference between the eigenvalue, corresponding to the nonlinear eigenvalue problem and the eigenvalue corresponding to the linear “unperturbed” problem. Within the small parameter method in case of eigenvalue problems in addition to the expansion of the eigenfunction \( f(\theta) \) in the series with respect to the small parameter \( \varepsilon \) the material nonlinearly index \( n \) is expanded:

\[
\lambda = \lambda_0 + \varepsilon, \quad f(\theta) = f_0(\theta) + \varepsilon f_1(\theta) + \varepsilon^2 f_2(\theta) + \ldots,
\]  

\[
n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \ldots,
\]  

where \( f_0(\theta) \) is the solution of the linear “unperturbed” problem. By substituting the asymptotic expansions (11) into the nonlinear differential equation (7) and the boundary conditions (9) and (10), we can obtain a sequence of boundary-value problems for linear differential equations with respect to the functions \( f_k(\theta) \):

\[
\varepsilon^0: \quad f^{(v)}_0 + 2(\lambda_0^2 + 1)f''_0 + (\lambda_0^2 - 1)^2 f_0 = 0,
\]

\[
f_0(\theta = 0) = 1, \quad f'_0(\theta = 0) = (\lambda_0 + 1)/\tan(M^p\pi/2),
\]

\[
f_0(\theta = \pi) = 0, \quad f'_0(\theta = \pi) = 0,
\]

\[
f_0(\theta = -\pi) = 0, \quad f'_0(\theta = -\pi) = 0, \quad f_0(\theta = 0) = 1,
\]

\[
f'_0(\theta = 0) = (\lambda_0 + 1)/\tan(M^p\pi/2).
\]  

\[
\varepsilon^1: \quad f^{(v)}_1 + 2(\lambda_0^2 + 1)f''_1 + (\lambda_0^2 - 1)^2 f_1 =
\]

\[
= -n_1 \left[ f_0(0) f_0'' + f_0 + 2w_0^2 f_0 + f_0'' g_0 + f_0'' f_0 + f_0'' g_0 - f_0'' f_0ight] -
\]

\[
= f_0'' f_0'' f_0'' - f_0'' f_0'' f_0'' - f_0'' f_0'' f_0'' - f_0'' f_0'' f_0'' -
\]

\[
f_0(\theta = 0) = 1, \quad f'_0(\theta = 0) = 1/\tan(M^p\pi/2),
\]
The solution of the ordinary linear differential equation with respect to function $f_0(\theta)$ (12), satisfying the boundary conditions of the absence of surface forces, is written as: for the normal loading of plate (for the mode I crack) $f''_0 = \beta \cos \alpha \theta - \alpha \cos \beta \theta$, $\alpha = \lambda_\alpha - 1$, $\beta = \lambda_\beta + 1$, for pure shear (cracks of the antiplane shear) $f''_0 = \sin \alpha \theta - \sin \beta \theta$, where the range of eigenvalues is determined from the solution of the characteristic equation $\sin 2\lambda \theta = 0$, from which $\lambda_\alpha = n/2$, where $m$ is a whole number. Under mixed loading of the specimen with a crack, because of the linearity of the "unperturbed" problem, the solution is a superposition of the symmetric and antisymmetric parts of the solution with respect to the continuation line of the defect

$$f_0 = C_1 (\beta \cos \alpha \theta - \alpha \cos \beta \theta) + C_2 (\sin \alpha \theta - \sin \beta \theta),$$

(15)

where the constants $C_1$ and $C_2$ are connected by the relation $M^p = 2 \arctg \left( \frac{\lambda_\alpha + 1}{\lambda_\beta / C_1 / C_2} \right) / \pi$. The zero-order problem has a nontrivial solution (15), therefore, the nonhomogeneous problems for the functions $f'_1(\theta)$ and $f'_2(\theta)$ (13), (14) will not have solutions, unless they will be satisfied with the solvability conditions [22, 23]. Thus, if the value of the parameter $\lambda_\alpha$ does not coincide with any of the eigenvalues of the homogeneous problem (that is, the
homogeneous problem has only a trivial solution), then the
inhomogeneous problem has a unique solution for any
continuous right-hand side $G_i(0)$ of the differential
equation with respect to the function $f_k(0), k > 0$. On the
other hand, if the parameter $\lambda_0$ is equal to some eigenvalue
of the homogeneous problem (that is, the homogeneous
problem has a nontrivial solution), the inhomogeneous
problem can be solved only under the condition [22, 23].

$$\int_{-\infty}^{\infty} G_i(0)u(0)d\theta = 0,$$  \hspace{1cm} (16)

provided that $G_i(0)$ function is orthogonal to the
eigenfunction $u(0)$, corresponding to eigenvalue $\lambda_0$. This
result provides the content of the theorem, which is usually
called Fredholm alternative [22, 23]. In order to find the
solvability condition for the boundary value problem
for the fourth-order equation (13), we can use the approach
used in [24-27] and show that the boundary value problem
(13) is self-adjoint, since the differential equations and
boundary conditions of the conjugate problem coincide with
the differential equation and the boundary conditions of the
homogeneous problem (12). Hence, $u(0) = f_0(0)$, where
function $f_0(0)$ is determined with the expression (15). The
condition of solvability of the boundary problem (13) is
written as $\int_{-\infty}^{\infty} G_i(0)f_0(0)d\theta = 0$ or in an explicit form

$$\int_{-\infty}^{\infty} \left\{ -n_1 \left( f_0(0) + \frac{2}{2}\omega_0 \right) \right\} (2g_0) +$$

$$+ n_2 \left( f_0 ' + \frac{3}{2}\omega_0 g_0 f_0 ' \right) - \frac{1}{2} f_0 \left[ (\lambda_0 - 1)(4\lambda_0 - 1) + 8\lambda_0 \right] \frac{1}{2} n_2 \left( \lambda_0 - 1 \right) \times$$

$$\times \left[ (\lambda_0 - 1)(4\lambda_0 + 1) + 8\lambda_0 \right] f_0(0)d\theta = 0. \hspace{1cm} (17)

The solvability condition of the boundary problem for
$f_0(0)$ function (17) makes it possible to find $n_1$ coefficient.
The obtained values of coefficient $n_1$ for different values of the
mixity parameter are given in Table 1. After the solution
of the boundary problem with respect to $f_0(0)$ function, one
can switch to integrating the equation for $f_2(0)$ function.
By a similar reasoning, the solvability condition of the
boundary problem for function $f_2(0)$ can be formulated,
and numerical values can be obtained for the following
coefficient of the asymptotic decomposition of the material
nonlinearity index. The computational values of $n_2$
coefficient are given in Table 1.

From the results of the calculations it can be seen that
mixed loading leads to a change of characteristics of the
stress field at the crack tip under plane stress state. It
was established earlier [24-26], that for the opening mode
cracks and antiplane shear $n_1 = (-1)^{k+1} / (\lambda_0 - 1)^{k+1}$, and the

<table>
<thead>
<tr>
<th>$M^0$</th>
<th>$n_1$</th>
<th>$n_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^0 = 0$</td>
<td>4.000000</td>
<td>8.000000</td>
</tr>
<tr>
<td>$M^0 = 0.05$</td>
<td>4.001766</td>
<td>7.999995</td>
</tr>
<tr>
<td>$M^0 = 0.1$</td>
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<td>7.999954</td>
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<tr>
<td>$M^0 = 0.2$</td>
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<td>7.978646</td>
</tr>
<tr>
<td>$M^0 = 0.3$</td>
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<td>7.941876</td>
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<tr>
<td>$M^0 = 0.4$</td>
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<td>7.804045</td>
</tr>
<tr>
<td>$M^0 = 0.5$</td>
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<td>7.749316</td>
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<tr>
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<td>7.958755</td>
</tr>
<tr>
<td>$M^0 = 0.95$</td>
<td>4.023759</td>
<td>7.999543</td>
</tr>
<tr>
<td>$M^0 = 1$</td>
<td>4.000000</td>
<td>8.000000</td>
</tr>
</tbody>
</table>

Asymptotic decomposition coefficients of the material
nonlinearity index

The asymptotic series for $n$ in expressions (11) can be summed,
since this expression is the sum of an infinitely decreasing
geometric progression. It is easy to find that $n = \lambda \lambda (\lambda + 1)$
or $\lambda = n (n + 1)$, which complies with the classical solution
of HRR. From the results of the calculations in Table 1 for
different values of the mixity parameter, it is clear that the
mixed loading leads to a change of characteristics of the
stress field at the crack tip under mixed-mode deformation
within the assumption of the plain stress state
implementation. Otherwise, the values for the coefficients
$n_1$ and $n_2$ would be equal to $n_1 = 4$ and $n_2 = 8$
respectively, for all the values of the mixed load parameter,
as it turns out in the case of the plane deformed state. In the
construction of a numerical solution of the problem related
to determining the stress-strain state in the vicinity of the
crack tip under plane deformed state, it is assumed that the
nature of the characteristics of the stress field coincides with
the singularity of the stress field of the HRR solution (that is,
the eigenvalue of the problem is known, and it is assumed a
priori that $\lambda = n (n + 1)$ and the assumption leading to a
continuous distribution of the radial stress is confirmed by
the solution obtained [6]). In the case of a plane stress state,
such an assumption, as shown by the asymptotic analysis, can not
be accepted, and it is necessary to search for the spectrum of
eigenvalues as part of the solution. In the next part the
numerical solution of the nonlinear eigenvalue problem under
consideration will be given.

Numerical Solution of the Nonlinear Eigenvalue
Problem. Eigenvalues and Eigenvalue Functions

As a rule, during the studies of the mixed loading, it is
assumed that the eigenvalue of the non-linear eigenvalue
problem under consideration is known $\lambda = n (n + 1)$.
However, this assumption leads to a discontinuous field of
radial stress, and, as an approximate analysis of the eigenvalues of the nonlinear eigenvalue problem under consideration shows that it cannot be used in the case of the plane stress state. For these reasons, we determine the eigenvalue of the problem, leading to a continuous field of radial stress. A similar approach was applied to find the entire range of eigenvalues in [15], where it was found that such a method of finding the eigenvalues leads to contours of the completely damaged material domain converging to some limiting contour. The numerical determination of eigenvalues is based on the following representations. In case of the mixed deformation, the assumptions of symmetry and antisymmetry can not be used, and it is necessary to find the solution of equation (7) at the domain $[-\pi, \pi]$. Under mixed loading during the numerical solition of equation (7), the domain of integration $[-\pi, \pi]$ can be divided into two sub-domains: $[0,\pi]$ and $[-\pi, 0]$. First, equation (7) is integrated at the domain $[0,\pi]$ and the two-point boundary value problem for equation (7) with boundary conditions (9) is reduced to the Cauchy problem

$$f(\theta = 0) = 1, \quad f'(\theta = 0) = (\lambda + 1) / \tan(M^* \pi / 2),$$

$$f^*(\theta = 0) = A_2, \quad f^*(\theta = 0) = A_3.$$  

The unknown constants $A_2$ and $A_3$ are defined in such a way that the boundary conditions on the upper edge of the crack are satisfied:

$$f(\theta = \pi) = 0, \quad f'(\theta = \pi) = 0. \quad (18)$$

After the constants $A_2$ and $A_3$ from the requirement of the fulfillment of the condition of the absence of surface forces, equation (7) is integrated at the domain $[-\pi, 0]$, for what the two-point boundary value problem for equation (7) with boundary conditions (10) is changed into the Cauchy problem

$$f(\theta = -\pi) = 0, \quad f'(\theta = -\pi) = 0,$$

$$f^*(\theta = -\pi) = B_2, \quad f^*(\theta = -\pi) = B_3.$$  

Unknown constants $B_2$ and $B_3$ are selected in such a way that they could fulfill the element's equilibrium condition, located at the ray $\theta = 0$. The equilibrium equation of this element requires continuity of the stress tensor components $q_{0\theta}(r, \theta)$ and $q_{\theta\theta}(r, \theta)$ at the ray $\theta = 0$, which implies the continuity of functions $f(\theta)$ and $f'(\theta)$ at $\theta = 0$ (and, consequently, the boundary conditions in (10)). That is why two unknown constants $B_2$ and $B_3$ are determined in such a way that the solution, found at the domain $[-\pi, 0]$, could satisfy the boundary conditions when $\theta = 0$. During the performance of the described numerical process, it is usually supposed that the eigenvalue is known and it equals the eigenvalue of HRR solution. If it is necessary to find other eigenvalues of the problem, different from $\lambda = n / (n+1)$, and, in general, the whole range of eigenvalues, the question arises on additional physical or mathematical considerations that should be involved in order to find the entire range of eigenvalues. If we consider, that $\lambda$ is the required value, then during the integration of equation (7) at the domain $[0, \pi]$, there are three unknown parameters, $\lambda$, $A_2$ and $A_3$, and there are only two conditions (18), from which they can be determined. It is obvious, that an additional condition is necessary to find eigenvalue $\lambda$. In order to determine the entire range of eigenvalues $\lambda$, one can analyze the behavior of the radial component of the stress tensor in the case of a plane deformed state [6, 15], and see that the radial component of the stress tensor is a continuous function of the polar angle for all the values of the mixity parameter and the hardening index of the material; while in the construction of the solution, the continuity of this component was not required (that is, before the construction of the numerical solution, we chose $\lambda = n / (n+1)$, and component $q_{0\theta}(r, \theta)$ turned out to be continuous for all the the values of the mixity parameter and the index of the material nonlinearity). In this connection, when finding the eigenvalues, it is necessary to require the continuity of the radial component of the stress tensor $q_{0\theta}(r, \theta)$ for $\theta = 0$. Therefore, further for the construction of new eigenvalues, an additional condition was imposed, i.e. the requirement of continuity of the stress tensor component $q_{0\theta}(r, \theta)$ for $\theta = 0$. The calculation results are given in Table 2-7, where new values of $\lambda$ and trial values $f^*(\theta = 0)$, $f^*(\theta = 0)$, $f^*(\theta = -\pi)$ and $f^*(\theta = -\pi)$ are collected for all the values of the the mixity parameter and practically important values of the material nonlinearity index $n$.

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<th>$M$</th>
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<th>$f^*(0)$</th>
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<th>$f^*(-\pi)$</th>
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Fig. 1. The angular distributions of the components of stress tensor ∂σ and ∂σ for different values of the mixity parameter for n = 3

The angular distributions of the components of stress and strain tensors obtained as a result of the numerical analysis for new eigenvalues are shown in Fig. 18. The lines of the equal values of stress intensity in the vicinity of the crack tip are shown in Fig. 9–13.

Fig. 2. The angular distributions of the component of stress tensor $\tilde{\sigma}_{rr}$ and stress intensity $\tilde{\sigma}_r$ for different values of the mixity parameter for $n = 3$

Fig. 3. The angular distributions of the strain tensor components $\tilde{\epsilon}_{\theta\theta}$ and $\tilde{\epsilon}_{rr}$ for different values of the mixity parameter for $n = 3$

Fig. 4. The angular distributions of the strain tensor component $\tilde{\epsilon}_{r\theta}$ and the intensity of deformation $\tilde{\varepsilon}_r$ for different values of the mixity parameter for $n = 3$
Fig. 5. The angular distributions of the stress tensor components $\bar{\sigma}_{\theta\theta}$ and $\bar{\sigma}_{r\theta}$ for different values of the mixity parameter for $n = 13$.

Fig. 6. The angular distributions of the stress tensor component $\bar{\sigma}_{rr}$ and stress intensity $\bar{\sigma}_e$ for different values of the mixity parameter for $n = 13$.

Fig. 7. The angular distributions of the strain tensor components $\bar{\varepsilon}_{\theta\theta}$ and $\bar{\varepsilon}_{rr}$ for different values of the mixity parameter for $n = 13$. 
Fig. 8. The angular distributions of the strain tensor components $\tilde{\varepsilon}_{\theta}$ and the strain intensity $\tilde{\varepsilon}_e$ for different values of the mixity parameter ($n = 13$).

Fig. 9. The stress intensity isolines

Fig. 10. Isolines of stress intensities $\sigma_e(r, \theta)$
Fig. 11. Isolines of stress intensities $\sigma_e(r,0)$

Fig. 12. Isolines of stress intensities $\sigma_e(r,0)$

Fig. 13. Isolines of stress intensities $\sigma_e(r,0)$
Results and Discussion

The asymptotic and numerical solution of the nonlinear eigenvalue problem is given, which follows from the problem of determining the stress-strain state in the vicinity of the crack tip under mixed loading in a full range of mixed deformation types from pure shearing-mode to normal opening-mode within the assumption of the plane stress state implementation. The solution is based on the expansion of mechanical quantities in a power series with respect to small degrees of distance from the crack tip in the vicinity of the crack tip. The expansion of the Airy stress function in a series with respect to eigenfunctions reduces the analysis of the stress-strain state at the crack tip to a nonlinear eigenvalue problem, one of the eigenvalues of which is well known and corresponds to the classical problem of nonlinear fracture mechanics, i.e. the HRR problem: \( \lambda = n/(n + 1) \). When studying the mixed deformation forms, according to the established tradition [6, 7, 8], it is assumed that under mixed loading this eigenvalue is the eigenvalue of the problem. For the case of a plane deformed state, this assumption turns out to be valid and is confirmed by the obtained solutions [6, 7, 8]. However, for a plane stress state, the hypothesis leads to a discontinuous field of radial stress on the crack extension line, which is not supported by the known finite element solutions, asymptotic studies, and experiments in fracture mechanics. In the present paper, based on the approaches of the asymptotic theory and on the numerical analysis of the problem, it is shown that under plane stress state, the hypothesis, that under mixed loading, the eigenvalue of the problem coincides with the eigenvalue of the HRR problem, is violated, so it is necessary to find new eigenvalues leading to continuous stress distributions. The asymptotic analysis of the eigenvalues, based on the methods of asymptotic analysis (small parameter method), made it possible to establish that the mixed loading of a plate with a crack leads to a new stress distribution different from the HRR solution. The article proposes the procedure for the numerical determination of the eigenvalue spectrum of the nonlinear eigenvalue problem. With the help of a numerical solution of the nonlinear eigenvalue problem, new eigenvalues are found that correspond to the angular distributions of the radial stress (continuous on the crack extension line). It is shown that for the special cases of the opening fracture mode and antiplane shear, the proposed technique leads to the well-known Hutchinson-Rice-Rosengren solution. It is necessary to mention the importance and relevance of the development of methods of asymptotic analysis and their applications to the solution of nonlinear eigenvalue problems in solid mechanics [20] and, in particular, in nonlinear fracture mechanics and continuum mechanics of damage [28, 32, 33], as in nonlinear fracture mechanics, the eigenfunction expansion method is one of the most frequently applied methods [25-32], that leads to nonlinear eigenvalue problems, which solution, in turn, requires attracting the developed asymptotic methods and numerical approaches, as well as their combinations. For example, in [26-32] it is shown that the damage accumulation near the crack tip leads to a change of the stress-strain state near the defect and to a weaker characteristic of field of stresses or to its complete elimination. The determination of a new radial asymptotics of the stress and strain fields (or strain rates) is reduced to nonlinear eigenproblems, which proper qualitative study entails the essence of the problem and provides the solution of the problem as a whole. The obtained solution also seems to be useful, when constructing self-similar solutions of the second order [33, 34] and nonlinear eigenvalue problems, related to them [15, 27, 29, 33]. The proposed method can be used to find the intermediate asymptotic self-similar representation of the stress field in a coupled (creep-damage) problem about the crack under material mixed loading with the power law equations of the theory of the settled creep. It is also worth mentioning that the class of nonlinear eigenvalue problems, occuring in nonlinear fracture mechanics, is important, as it is necessary to construct multiscale, multilevel models [31, 33, 34], according to which in the vicinity of the crack tip it is necessary to consider a set of domains with a domination of different asymptotics of the stress field, and perform the process of the asymptotic conjugation of the obtained solutions. An accurate construction of all the intermediate zones with either asymptotics or conjugation requires understanding the complete range of eigenvalues, and, presumably, these problems have not been solved yet.

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References


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