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ON THE GEOMETRICAL THEORY OF STATIONARY TURBULENT FLOW OF BLOOD

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Abstract. The consideration is being given to the turbulent flow of blood in a part of a vessel. Therefore flow of blood can be simulated as flow of a liquid in the three-dimensional Euclidian space. In the beginning, it is shown that the normal congruence of lines to plane allocation in the Euclidian space represents circular helixes. Further it is proven that to every turbulent flow of blood, the streamlines of which represent circular helixes, there corresponds plane allocation.

Key words: flow of blood, turbulence, congruence, plane allocation, circular helixes

Introduction

The modelling of the human cardiovascular system work puts a set of problems. For overcoming many of them it is necessary to know the trajectories of moving blood particles. At stationary conditions the trajectory of a particle represents a streamline of a vector field characterising the velocities of blood particles which move in the cardiovascular system.

The case of turbulent flow of blood was considered by the authors in [1], where this flow was modelled as flow of liquid along geodesic lines of the subprojective space. The subprojective space represents the special Riemannian space [2]. The necessary mathematical technique to describe blood flow and also to derive a geometrical pattern of blood stream was given. The basis for such an approach comes from the simple fact that if Reynolds numbers describing turbulent flow are large, the main role is played by inertia and geodesic Riemannian spaces are nothing but trajectories of movement on inertia [3].

In this article, turbulent flow of blood in a vessel is considered. Therefore blood flow can be modelled as flow of liquid in the three-dimensional Euclidean space. In the beginning it is shown that the normal congruence of lines to the plane allocation in the Euclidean space represents circular helixes. Further it is proved that the plane allocation corresponds to every turbulent blood flow, the streamlines of which represent circular helixes.

Normal congruence of lines to plane allocation in E^3

In the three-dimensional Euclidean space E^3, let us consider an allocation Δ^2 given in some area Ω. Then (x, ξ) is a unit of this allocation consisting of a point $x \in \Omega$ and a plane $\xi = \xi(x)$ passing through it. The integrated line l of the given allocation is named asymptotic, if in each point $x \in l$ the principal curvature-plane $\pi(x)$ of this line belongs to a plane $\xi(x)$. The allocation Δ^2 is named plane, if all directions inherent to the units are asymptotic.

Let us join the frame $\{x, \mathbf{e}_i, \mathbf{e}_3\}$ to the unit (x, ξ) of the allocation Δ^2 so that $\mathbf{e}_i \in \xi$, $\mathbf{e}_3$ be perpendicular to ξ and $|\mathbf{e}_i| = 1$. The index i, as well as all the indexes designated by small characters of the Latin alphabet, accepts values 1, 2. Then the equations of transition of this frame look like

$$dx = \omega^A \mathbf{e}_A, \quad d\mathbf{e}_A = \omega^B{}_A \mathbf{e}_B, \tag{1}$$

where A, B = 1, 2, 3.

The main set of equations defining the allocation Δ^2 may be written as [4]

$$\omega^3{}_i = \Lambda_{ij}\, \omega^j + \Lambda_i\, \omega^3, \tag{2}$$

where Λ_{ij} is the main tensor of the allocation, and Λ_i is the covariant vector of the given allocation. For simplification of further evaluations we canonise the given frame in such a manner that $\Lambda_1 = 0$ and $\Lambda_2 \neq 0$. Then for the plane allocation ($\Lambda_{ij} = -\Lambda_{ji}$) equation (2) is rewritten as

$$\omega^3{}_1 = \Lambda_{12}\, \omega^2, \quad \omega^3{}_2 = \Lambda_{21}\, \omega^1 + \Lambda_2\, \omega^3. \tag{3}$$

The differential forms ω^A and $\omega^B{}_A$ included in equations (1) satisfy the equations of the frame of the Euclidean space:

$$D\omega^i = \omega^j \wedge \omega^i{}_j, \quad D\omega^i{}_j = \omega^k{}_j \wedge \omega^i{}_k. \tag{4}$$

By differentiating externally the equations (3) and using the equations of the frame (4), we note

$$(\nabla\Lambda_{ij} - \Lambda_i\, \omega^3{}_j) \wedge \omega^j + (\nabla\Lambda_i - \Lambda_{ij}\, \omega^j{}_3) \wedge \omega^3 = 0, \tag{5}$$

where $\nabla\Lambda_{ij} = d\Lambda_{ij} - \Lambda_{ik}\, \omega^k{}_j - \Lambda_{kj}\, \omega^k{}_i$, $\nabla\Lambda_i = d\Lambda_i - \Lambda_j\, \omega^j{}_i$.

After application to equalities (5) the Cartan lemma, we have

$$\nabla\Lambda_{ij} - \Lambda_i\omega^3_j = \mu_{ijk}\omega^k + \mu_{ij}\omega^3;\ \nabla\Lambda_i - \Lambda_{ij}\omega^j_3 = \mu_{ij}\omega^j + \mu_i\omega^3, \tag{6}$$

where $\mu_{ijk} = \mu_{ikj}$.

It follows from (6) that

$$\mu_{121} = -\mu_{211},\ \mu_{122} + \mu_{212} = -\Lambda_2\Lambda_{12},\ \mu_{12} = -\mu_{21},$$
$$\mu_{111} = \mu_{112} = \mu_{11} = 0,\ \mu_{221} = -\Lambda_2\Lambda_{21},\ \mu_{222} = 0,\ \mu_{22} = -\Lambda_2^2, \tag{7}$$

and also

$$\omega_1^2 = -\frac{\Lambda_{21}^2}{\Lambda_2}\omega^1 - \frac{\mu_{12}}{\Lambda_2}\omega^2 + \left(\Lambda_{12} - \frac{\mu_1}{\Lambda_2}\right)\omega^3. \tag{8}$$

Then the equations (6) look like:

$$d\Lambda_{21} = -2\Lambda_2\Lambda_{21}\omega^2 + \mu_{21}\omega^3,$$
$$d\Lambda_2 = \mu_{21}\omega^1 + (\Lambda_{21}^2 - \Lambda_2^2)\omega^2 + \mu_2\omega^3. \tag{9}$$

By differentiating externally equations (9) and also multiplying externally by ω^3, we obtain that $\mu_{21} = 0$ and $\mu_2 = 0$. Then equalities (9) are written as

$$d\Lambda_{21} = -2\Lambda_2\Lambda_{21}\omega^2,$$
$$d\Lambda_2 = (\Lambda_{21}^2 - \Lambda_2^2)\omega^2. \tag{10}$$

It is seen from (10) that Λ_2 = const and Λ_{21} = const. We have proved the following theorem.

Theorem 1. For the plane allocation in E^3 the main tensor Λ_{ij} and the covariant vector Λ_i have the constant values in area Ω.

The normal congruence of lines to the plane allocation has the vector $\mathbf{e}_3$ as a tangential vector, i.e. the given congruence may be understood as the integrated lines of a field of vector $\mathbf{e}_3$. Then the curvature of an integrated line of the vector field is calculated in the following way:

$$k = |\Lambda_2|\ (\text{mod}\ \omega^i = 0). \tag{11}$$

Therefore since Λ_2 = const, then k = const.

The torsion of this line is also easily calculated: $\chi = \Lambda_{21}$. In view of constancy of Λ_{21}, we conclude that χ = const. We have proved the theorem.

Theorem 2. The normal congruence of lines to the plane allocation in E^3 consists of circular helixes.

Case of turbulent flow of blood when the streamlines are circular helixes

At laminar flow of blood in a vessel the allocation of velocities on the cross-section of a vessel has a parabolic profile. In case of turbulent flow, the profile of the velocity allocation becomes less prolate. It is a corollary of the fact that liquid goes not only in a parallel way to the axis of a vessel, but also across a vessel. So the average velocity of blood flow appears to be almost constant within the whole cross-section of a vessel and only in a narrow layer near the vessel walls, due to adhesion, the velocity begins to be decreased. In the given situation, the velocity of blood at turbulent flow may be considered as independent from the radius of the cylinder, in which the blood goes and on which the circular helix being a streamline is placed.

In some rectangular frame (O, **i, j, k**), whose centre lays on the axis of the appropriate cylinder, equation of a circular helix can be written as:

$$x = \rho \cos \phi, \ \ y = \rho \sin \phi, \ \ z = v\, \phi, \tag{12}$$

where $r = \sqrt{x^2 + y^2}$, v is the blood velocity which is not the function of the radius of the cylinder or distance from the axis of a vessel to the cylinder on which the given circular helix is placed.

The tangential vector **A** to a circular helix looks like: $\mathbf{A}(x_\phi, y_\phi, z_\phi)$ or $\mathbf{A}(-y, x, v)$. Then a vector **a** (i.e. the single vector of the field **A**) has the following coordinates:

$$\mathbf{a}\left(\frac{-y}{\sqrt{\rho^2+v^2}}, \frac{x}{\sqrt{\rho^2+v^2}}, \frac{v}{\sqrt{\rho^2+v^2}}\right).$$

It is easily seen that the average curvature of the field of vectors **a**, defined as $2H = -\operatorname{div} \mathbf{a}$, is equal to zero. To calculate the total curvature K of the field of vector **a** let us express K with the help of components of vector **A**: $\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \lambda\, \mathbf{a}$, where $\lambda = \sqrt{A_1^2 + A_2^2 + A_3^2} = \sqrt{x^2+y^2+z^2}$. Consider vector **P** $\{(\mathbf{a}_y, \mathbf{a}_z, \mathbf{a}), (\mathbf{a}_z, \mathbf{a}_x, \mathbf{a}), (\mathbf{a}_x, \mathbf{a}_y, \mathbf{a})\}$, where the parentheses designate mixed products of the appropriate vectors. Then, according to [5], $K = (\mathbf{P}, \mathbf{a})$. After the calculations, we obtain:

$$K = \frac{-v(\rho^2 v' - v)}{(\rho^2+v^2)^2} = \frac{v^2}{(\rho^2+v^2)^2}, \tag{13}$$

as $v' = 0$ in view of the above imposed condition about the constancy of velocity v.

Let us calculate the magnitude of nonholonomy of the field **A** [6]: $(\mathbf{A}, \operatorname{rot} \mathbf{A}) = 2v$. Then

$$\operatorname{rot} \mathbf{a} = \frac{2}{\sqrt{\rho^2+v^2}}\, \mathbf{k} \quad \text{and} \quad (\mathbf{a}, \operatorname{rot} \mathbf{a}) = \frac{2v}{\rho^2+v^2}. \tag{14}$$

Due to $v \neq 0$ practically everywhere in the considered vessel, it is clear from (14) that field **a** is nonholonomic and there is no set of surfaces such that in each point the vector of the field is directed towards the normal of the surface of the set passing through this point. So, the given field of vector is orthogonal to the allocation Δ^2 which is nonholonomic or quite integrable.

The direction of d**x**, belonging to the allocation Δ^2 and orthogonal to the field of vector **a**, is asymptotic, if the normal curvature of this field in the direction of d**x** equals zero [5]. From here it follows that the asymptotic direction is defined from the vector equation:

$$[\mathbf{a}, d\mathbf{a}] = \mu\, d\mathbf{x}, \tag{15}$$

where $\mu \in R$.

We can easily find an equation, from which μ is determined:

$$\mu^2 + (\mathbf{a}, \mathrm{rot}\,\mathbf{a})\,\mu + K = 0. \tag{16}$$

The solutions of the equation (16) are as follows:

$$\mu_{1,2} = \frac{-(\mathbf{a}, rot\,\mathbf{a}) \pm \sqrt{(\mathbf{a}, rot\,\mathbf{a})^2 - 4K}}{2}. \tag{17}$$

At constancy of the velocity, it is easily shown that $(\mathbf{a}, \mathrm{rot}\,\mathbf{a})^2 - 4K = 0$, that is $?_1 = ?_2$. Then, taking into account (15), any direction orthogonal to field **a** and belonging to the allocation Δ^2 is asymptotic, that is the allocation Δ^2 is plane.

We have proved the following theorem.

Theorem 3. If the turbulent flow of blood has circular helixes as streamlines, the given circular helixes are orthogonal to the plane allocation.

Note. Theorem 3 can be formulated as follows: the turbulent flow of blood, whose streamlines are screw lines, is in accordance with the plane allocation and geometry of such blood flow may be considered as geometry of normal congruence of lines to the plane allocation.

The last reasoning is correct for all the points of a vessel, except for a layer directly related to the vessel wall.

Taking into account Theorem 2 and Theorem 3 it is easily proven the following theorem.

Theorem 4. The stationary flow of blood with streamlines as circular helixes is turbulent if and only if the velocity vector is orthogonal to the plane allocation.

Conclusions

It is possible to carry out the study of turbulent flow of blood based on the geometry of integrated lines of the field of velocity vector by considering of the geometry of the normal congruence of lines to the allocation and the geometry of such allocations. In this article, a case of the plane allocation was considered. Such an approach, in many aspects, facilitates the study of complicated turbulent flow of blood which takes place in the heart, an ejection of blood from the heart, flow in the places of branching of vessels and in pathological vessels, and also in those cases, when Reynolds number exceeds the critical value.

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К ГЕОМЕТРИЧЕСКОЙ ТЕОРИИ СТАЦИОНАРНОГО ТУРБУЛЕНТНОГО ДВИЖЕНИЯ КРОВИ

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В данной работе рассматривается турбулентное движение крови в части сосуда, благодаря чему рассмотрение ведется в трехмерном евклидовом пространстве. Вначале показывается, что нормальная конгруэнция линий к плоскому распределению в евклидовом пространстве состоит из винтовых линий. Далее показывается, что каждому турбулентному движению крови, линии тока которого представляют собой винтовые линии, в данных условиях, соответствует плоское распределение. Библ. 6.

Ключевые слова: течение крови, турбулентность, конгруэнция, плоское распределение, винтовая линия

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