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# A CONFORMAL MAPPING SOLUTION OF THE HUMAN BONE TWISTING PROBLEM

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**Abstract.** Twisting problem of human bone may be solved using problem of Neumann or problem of Dirichlet by determining a function that is harmonic in a region of bone cross section and which assumes prescribed values on the boundary of bone cross section. It is very difficult to find this function. Therefore in this paper we show the method of solution of twisting problem of human bone by means of conformal mapping. We find formulas to obtain tangential stress distributions in arbitrary bone cross sections and resultant twisting moment.

**Key words:** conformal mapping, arbitrary bone cross sections

## 1. Introduction

In this paper, we consider the existence of the twisting moment in human joint produced by the twisting load applied to the human bone. For example, human thigh is connected with the hip joint and is loaded by the trunk forces, muscle forces, twisting moment, various couple of forces. Cross section of the bone has circular shape ravely (see Fig. 1). Therefore in this paper we solve the problem of bone twisting for irregular, non-circular shapes by means of conformal mapping.

Let the functional relationship:

$$z = w(\varsigma), \tag{1}$$

set up a correspondence between the point $\varsigma = \xi + i\eta$ in the complex $\varsigma$-plane and the point $z = x + iy$ in the complex *z*-plane.

If $z = w(\varsigma)$ is analytic in the bone cross section $R^*$ of the $\varsigma$-plane, then the totality of values $z$ belongs to the some region $R$ of the *z*-plane and is said that the cross section $R^*$ of the bone is mapped into the region $R$ and $R$ into $R^*$ by the mapping function $w(\varsigma)$ and reciprocal function $w^{-1}(z)$, respectively (see Fig. 2).

If $C_0^*$ is some curve drawn in the bone cross section $R^*$ and the point $\varsigma$ is allowed to move along $C_0^*$, then the corresponding point *z* will trace a curve $C_0$ in the z-plane and $C_0$ is called the mapping of $C_0^*$. If $C_1^*$ and $C_2^*$ are two curves in the $\varsigma$-plane that intersect at an angle $\delta$, then the corresponding curves $C_1$ and $C_2$ in the *z*-plane also intersect at the angle $\delta$, since the tangents to these curves are rotated through the same angle.

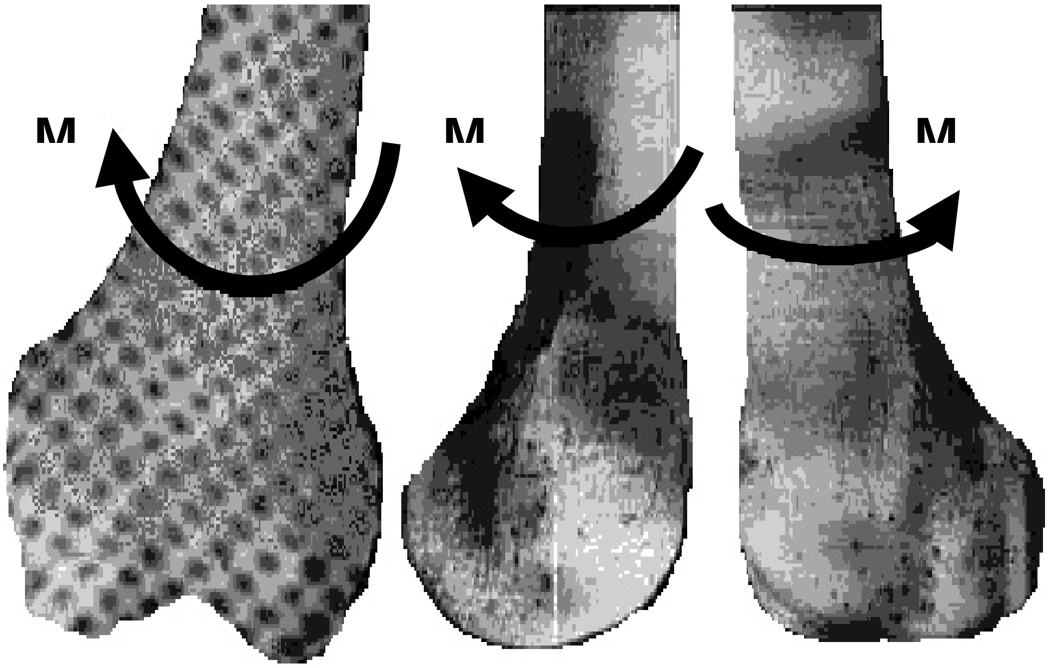


Fig. 1. Loading model of the lower end of the thigh human bones with twisting moment.

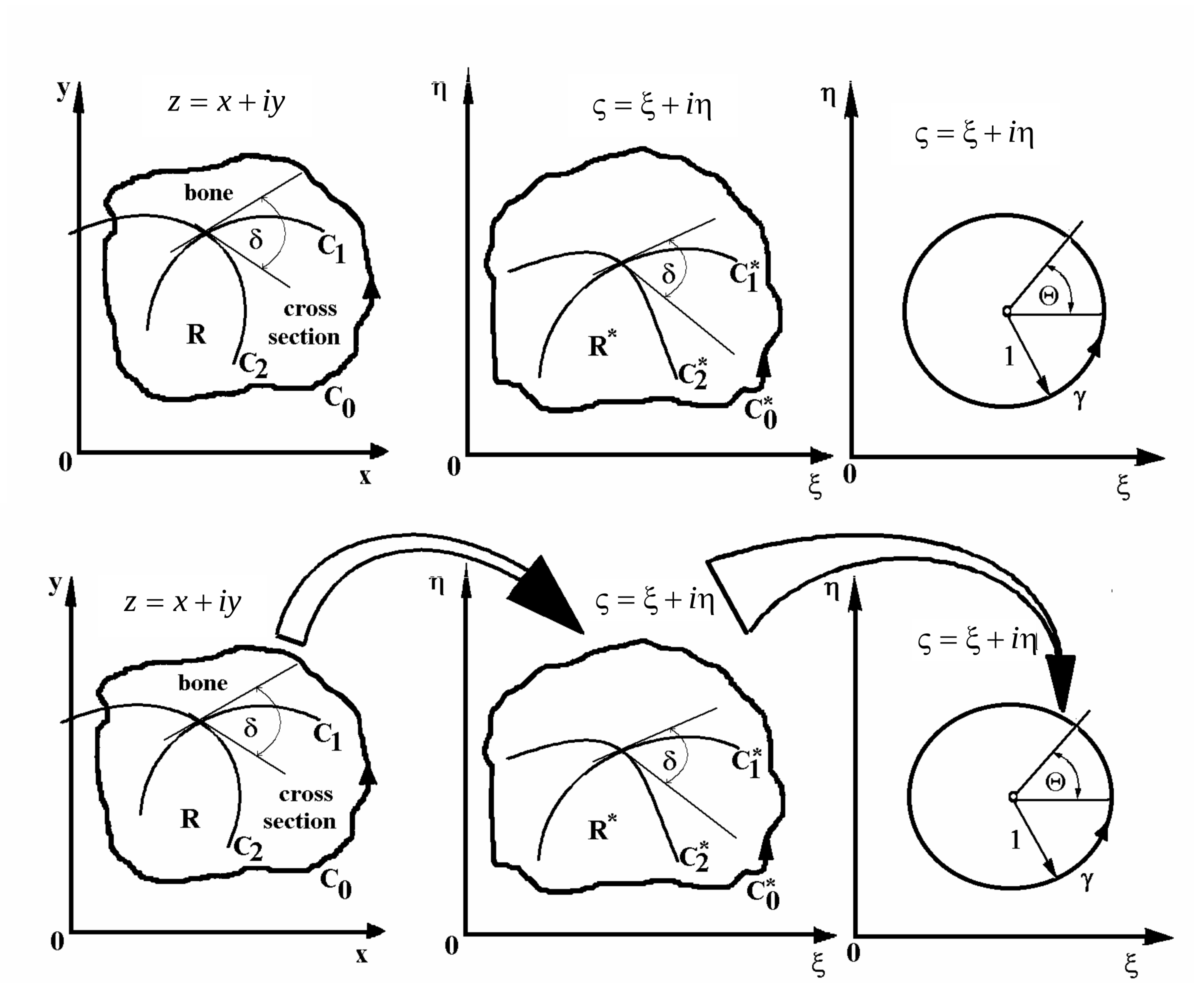


Fig. 2. Two conformal transformations of human bone cross section $R$ on region $R^*$ and on the unit circle region with boundary $\gamma$.

A transformation that preserves angles is called conformal [1-3]. The mapping performed by an analytic function $w(\varsigma)$ is conformal at all points of the $\varsigma$-plane where $w'(\varsigma) \neq 0$.

## 2. Solution of the bone twisting problem for an arbitrary mapping function

We assume that twisting function $\phi(x, y)$ be combined with the conjugate function $\psi(x,y)$ to form the following function:

$$F(z) = \phi(x, y) + i\,\psi(x, y), \tag{2}$$

where $z = x + iy$. Function $F(z)$ is analytic in the simple connected bone cross section $R$ [3]. Therefore there exists function (1) which maps bone cross section $R$ on the unit circle $|\varsigma| \leq 1$, where the boundary of the unit circle $|\varsigma| = 1$ will be denoted by the letter $\gamma$ and the points on the boundary $\gamma$ by:

$$\sigma = e^{i\theta},\ 0 \leq \theta < 2\pi. \tag{3}$$

We put equation (1) into expression (2) and hence function (2) can be expressed in terms of variable $\varsigma$ as follows:

$$\phi(x, y) + i\,\psi(x, y) = F[w(\varsigma)] \equiv f(\varsigma). \tag{4}$$

The imaginary part of function $f(\varsigma)$, i.e. harmonic function $\psi$ in Dirichlet twisting problem, satisfies on the boundary $C_0$ of region $R$ or on $\gamma$ the good known conditions, which are now write in the following complex form [4]:

$$\psi(x, y) = \frac{1}{2}\left(x^2 + y^2\right) = \frac{1}{2}(x + iy)(x - iy) = \frac{1}{2} z\bar{z}. \tag{5}$$

We put (1) into (5), hence on the unit circle $\gamma$ we obtain:

$$\psi(x, y) = \frac{1}{2} w(\varsigma)\overline{w(\varsigma)} = \frac{1}{2} w(\varsigma)\overline{w}(\bar{\varsigma}) \tag{6}$$

for boundary circle $|\varsigma| = 1$, i.e. $\gamma$.

It follows from (4) that the function $\psi$ is the real part of the following complex expression:

$$\frac{1}{i} f(\varsigma) = \psi(x, y) + \frac{1}{i}\phi(x, y) = \psi - i\phi. \tag{7}$$

We use Schwarz and Poisson theorems [3-4] for expressions (6)-(7), hence we obtain:

$$\begin{aligned} \frac{1}{i} f(\varsigma) &= \frac{1}{\pi i}\int_\gamma \frac{1}{2}\left[\frac{1}{i} f(\varsigma) + \overline{\frac{1}{i} f(\varsigma)}\right]_{\varsigma=\sigma} \frac{1}{\sigma - \varsigma} d\sigma - a_o + b_o = \\ &= \frac{1}{\pi i}\int_\gamma \frac{1}{2}\left[(\psi - i\phi) + \overline{(\psi - i\phi)}\right]_{\varsigma=\sigma} \frac{1}{\sigma - \varsigma} d\sigma - a_o + ib_o = \frac{1}{\pi i}\int_\gamma \frac{(\psi)_{\varsigma=\sigma}}{\sigma - \varsigma} d\sigma - a_o + ib = \\ &= \frac{1}{\pi i}\int_\gamma \left[\frac{1}{2} w(\varsigma)\overline{w(\varsigma)}\right]_{\varsigma=\sigma} \frac{d\sigma}{\sigma - \varsigma} - a_o + ib_o = \frac{1}{\pi i}\int_\gamma \frac{1}{2} w(\sigma)\overline{w}(\bar{\sigma}) \frac{d\sigma}{\sigma - \varsigma} - a_o + ib_o, \end{aligned} \tag{8}$$

where $a_o$ and $b_o$ are constants and

$$\bar{\sigma} = \overline{e^{i\theta}} = e^{-i\theta} = \frac{1}{\sigma}. \tag{9}$$

Noting expressions (4), (9), then the integral (8) can be written as:

$$f(\varsigma) = \phi + i\,\psi = \frac{1}{2\pi}\int_\gamma \frac{w(\sigma)}{\sigma - \varsigma}\overline{w}\left(\frac{1}{\sigma}\right) d\sigma + const. \tag{10}$$

The resultant twisting moment of the external forces applied to the end of the bone has the form [4-6]:

$$M = \alpha D, \tag{11}$$

where $\alpha$ is the angle of twist per unit bone length. Rigidity of the bone subjected to the twisting may be written after differentiation in the following form [4]:

$$D = GJ_o + GD_o, \tag{12}$$

where

$$J_o = \iint_R \left(x^2 + y^2\right) dxdy = \iint_R \left[\frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial x}\left(xy^2\right)\right] dxdy, \tag{13}$$

$$D_o = \iint_R \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right) dxdy = \iint_R \left[\frac{\partial}{\partial y}(x\phi) - \frac{\partial}{\partial x}(y\phi)\right] dxdy, \tag{14}$$

$G$ is the shear modulus of the bone, $R$ is the area of bone cross section bounded by $C_o$.

Now we must show expression (14) in terms of function $f(\varsigma)$. An application of Green theorem [1, 3] to the expression (14) gives:

$$D_o = \iint_R \left[\frac{\partial}{\partial y}(x\phi) - \frac{\partial}{\partial x}(y\phi)\right] dxdy - \int_{C_o} \phi \cdot (xdx + ydy). \tag{15}$$

We use the polar co-ordinates and simple partial derivatives in curvilinear integral (15), hence

$$D_o = -\int_{C_o} \phi \cdot (xdx + ydy) = -\int_{C_o} \phi \cdot \left[\frac{\partial}{\partial x}\frac{1}{2}\left(x^2 + y^2\right)dx + \frac{\partial}{\partial y}\frac{1}{2}\left(x^2 + y^2\right)dy\right] = -\int_{C_o} \phi \cdot \left(\frac{1}{2}r^2\right), \tag{16}$$

where $x^2 + y^2 = r^2$.

Let the relationship:

$$z = w(\varsigma) = w(\sigma), \tag{17}$$

set up a correspondence between $z = x + iy$ of the complex $z$-plane and the unit circle bounded by $\gamma$ with point $\sigma = e^{i\theta}$. In this case, we have:

$$r^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z} = w(\sigma)\overline{w(\sigma)} = w(\sigma)\bar{w}(\bar{\sigma}), \tag{18}$$

and by virtue of expression (4) for $\varsigma = \sigma$ we have:

$$\phi = \frac{(\phi + i\psi) + (\phi - i\psi)}{2} = \frac{(\phi + i\psi) + \overline{(\phi + i\psi)}}{2} = \frac{1}{2}\left[f(\sigma) + \overline{f(\sigma)}\right] = \frac{1}{2}\left[f(\sigma) + \bar{f}(\bar{\sigma})\right]. \tag{19}$$

If we put expressions (18) and (19) in formula (16), we obtain:

$$D_o = -\frac{1}{4}\int_\gamma \left[f(\sigma) + \bar{f}(\bar{\sigma})\right] d\left[w(\sigma)\bar{w}(\bar{\sigma})\right]. \tag{20}$$

Now we must show polar moment of inertia of bone cross section $R$ in terms of functions $f(\varsigma)$. An application of Green theorem [1, 3] to the expression (13) gives polar moment of inertia in the following form:

$$J_o = \iint_R \left[\frac{\partial}{\partial y}\left(x^2 y\right) + \frac{\partial}{\partial x}\left(xy^2\right)\right] dxdy = -\int_{C_o} xy(xdx - ydy). \tag{21}$$

By virtue of dependence $z = x + iy$ it follows that:

$$x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}), \quad dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = \frac{1}{2i}(dz - d\bar{z}). \tag{22}$$

We put dependencies (22) in the right hand of equation (21), hence we obtain:

$$J_o = -\int_{C_o} \frac{1}{4}\left[z^2 - (\bar{z})^2\right]\left[\frac{1}{4}(z+\bar{z})(dz + d\bar{z}) + \frac{1}{4}(z-\bar{z})(dz - d\bar{z})\right]. \quad (23)$$

We multiply the terms in integral (23), thus polar moment of inertia of bone cross section has the following form:

$$J_o = -\frac{1}{8i}\int_{C_o}\left[z^2 - (\bar{z})^2\right](zdz + \bar{z}d\bar{z}) = -\frac{1}{8i}\int_{C_o} z^3 dz + \frac{1}{8i}\int_{C_o}(\bar{z})^3 d\bar{z} + \frac{1}{8i}\int_{C_o}(\bar{z})^2 zdz - \frac{1}{8i}\int_{C_o} z^2(\bar{z})d\bar{z}. \quad (24)$$

Bone cross section $R$ is bounded by the closed round $C_o$ and $z^3$, $(\bar{z})^3$ are the analytic functions, hence by virtue of Cauchy theorem [1, 3]:

$$\frac{1}{8i}\int_{C_o} z^3 dz = 0, \quad \frac{1}{8i}\int_{C_o}(\bar{z})^3 d\bar{z} = 0. \quad (25)$$

Now we integrate the following expression by parts:

$$-\frac{1}{8i}\int_{C_o} z^2(\bar{z})d\bar{z} = -\frac{1}{8i}\int_{C_o} z^2 d\left(\frac{1}{2}(\bar{z})^2\right) = -\frac{1}{8i}\left\{\frac{1}{2}\left[(z\bar{z})^2\right]_{z\in C_o} - \int_{C_o} z(\bar{z})^2 dz\right\}. \quad (26)$$

From Dirichlet conditions [4, 5] for twisting theory it follows that:

$$\frac{1}{2}\left[(z\bar{z})^2\right]_{z\in C_o} = \frac{1}{2}\left[(x+iy)^2\left(\overline{x+iy}\right)^2\right]_{z\in C_o} = \frac{1}{2}\left[\left(x^2 + y^2\right)^2\right]_{z\in C_o} = 0. \quad (27)$$

If we put expression (27) in formula (26), we obtain the following dependence:

$$-\frac{1}{8i}\int_{C_o} z^2(\bar{z})d\bar{z} = \frac{1}{8i}\int_{C_o} z(\bar{z})^2 dz. \quad (28)$$

We put formulae (25)–(28) in expression (24), hence polar moment of inertia of bone cross section has the following form:

$$J_o = \frac{1}{4i}\int_{C_o} z(\bar{z})^2 dz. \quad (29)$$

Let the relationship (17) set up the correspondence between complex $z$-plane and the unit circle bounded by $\gamma$ with points $\sigma = e^{i\theta}$. In this case, polar moment of inertia of bone cross section is:

$$J_o = -\frac{1}{4}i\int_{\gamma}\left[\bar{w}(\bar{\sigma})\right]^2 w(\sigma)dw(\sigma). \quad (30)$$

We put (12), (20), (30) in (11), hence the resultant twisting moment obtains the following final form:

$$M = -\frac{1}{4}\alpha G\int_{\gamma}\left[f(\sigma) + \bar{f}(\bar{\sigma})\right]d\left[w(\sigma)\bar{w}(\bar{\sigma})\right] - \frac{1}{4}i\alpha G\int_{\gamma}\left[\bar{w}(\bar{\sigma})\right]^2 w(\sigma)dw(\sigma), \quad (31)$$

where we remember that function $w$ defined by formula (1) maps points $\varsigma = \xi + i\eta$ in the complex $\varsigma$-plane into points $z = x + iy$ in complex $z$-plane and in special case maps area of the unit circle bounded by $\gamma$ with points $\sigma = e^{i\theta}$ into region $R$ in complex $z$-plane. Function $f$ is defined by formula (10) by means of function $w$.

Now we go to determine shear stresses [5]:

$$\tau_{yz} = G\alpha\left(\frac{\partial\phi}{\partial y} + x\right), \; \tau_{xz} = G\alpha\left(\frac{\partial\phi}{\partial x} - y\right), \; \tau_{xy} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0 \quad (32)$$

in terms of function $f(\varsigma)$.

We use stresses (32) to combine the following complex function:

$$\tau_{zx} - i\tau_{zy} = G\alpha\left[\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y} - (y + ix)\right]. \tag{33}$$

By virtue of Cauchy- Riemann dependence [3]:

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}, \tag{34}$$

formula (33) has the following form:

$$\tau_{zx} - i\tau_{zy} = G\alpha\left[\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} - i(x - iy)\right]. \tag{35}$$

The first derivative of analytical function $F(z)$ {see equation (2)} with respect to the complex variable $z = x + iy$ has the following form [3]:

$$\frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}. \tag{36}$$

Hence complex shear stress function (35) may be written in the following form:

$$\tau_{zx} - i\tau_{zy} = G\alpha\left[\frac{dF}{dz} - i\bar{z}\right]. \tag{37}$$

Taking into account that:

$$F(z) = F[w(\varsigma)] \equiv f(\varsigma), \tag{38}$$

the function $F$ has the following derivative with respect to the complex variable $z$:

$$\frac{dF}{dz} = \frac{df}{d\varsigma}\frac{d\varsigma}{dz} = \frac{df}{d\varsigma}\frac{1}{\frac{dz}{d\varsigma}} = \frac{df}{d\varsigma}\frac{1}{\frac{dw}{d\varsigma}} \quad \text{and} \quad \bar{z} = \bar{w}(\bar{\varsigma}). \tag{39}$$

We put expressions (39) into (37), hence shear stress function in bone cross section has the following final form:

$$\tau_{zx} - i\tau_{zy} = G\alpha\left[\frac{\frac{df}{d\varsigma}}{\frac{dw}{d\varsigma}} - i\bar{w}(\bar{\varsigma})\right]. \tag{40}$$

**3. Solution of the bone twisting problem for power series mapping function**

The mapping function is written in the form:

$$z = w(\varsigma) = \sum_{n=0}^{\infty} a_n \varsigma^n . \tag{41}$$

Now we give formal solution of the twisting problem in terms of coefficients $a_n$.

From formula (41) it follows that:

$$w(\sigma)\bar{w}(\bar{\sigma}) = w(\sigma)\bar{w}\left(\frac{1}{\sigma}\right) = \left(\sum_{m=0}^{\infty} a_m \sigma^m\right)\left(\sum_{n=0}^{\infty} \bar{a}_n \sigma^{-n}\right). \tag{42}$$

We multiply two infinite series in formula (42) using Cauchy method, hence we obtain:

$$w(\sigma)\bar{w}\left(\frac{1}{\sigma}\right) = \sum_{n=0}^{\infty} b_n \sigma^n + \sum_{n=1}^{\infty} \bar{b}_n \sigma^{-n}, \tag{43}$$

where for $n$ = 0, 1, 2, ... we have:

$$b_n = \sum_{r=0}^{\infty} a_{n+r}\bar{a}_r , \; \bar{b}_n = \sum_{r=0}^{\infty} \bar{a}_{n+r} a_r . \tag{44}$$

We put formula (43) in expression (10), hence:

$$f(\varsigma)=\frac{1}{2\pi}\int_{\gamma}\left[\sum_{n=0}^{\infty}b_n\sigma^n+\sum_{n=1}^{\infty}\bar{b}_n\sigma^{-n}\right]\frac{d\sigma}{\sigma-\varsigma}\equiv A_1+A_2\,. \tag{45}$$

Cauchy integral formula [3, 4] gives:

$$A_1\equiv\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{\gamma}\frac{b_n\sigma^n}{\sigma-\varsigma}d\sigma=i\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\gamma}\frac{b_n\sigma^n}{\sigma-\varsigma}d\sigma=i\sum_{n=0}^{\infty}\left(b_n\sigma^n\right)_{\sigma=\varsigma}=i\sum_{n=0}^{\infty}b_n\varsigma^n\,. \tag{46}$$

Since $|\varsigma|<1$, we have:

$$A_2\equiv\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{\gamma}\frac{\bar{b}_n\sigma^{-n}}{\sigma-\varsigma}d\sigma=\frac{1}{2\pi}\int_{\gamma}\sum_{n=1}^{\infty}\bar{b}_n\sigma^{-n}\left(\frac{1}{\sigma}+\frac{\varsigma}{\sigma^2}+\frac{\varsigma^2}{\sigma^3}+.....\right)d\sigma=$$

$$=\frac{1}{2\pi}\sum_{n=1}^{\infty}\bar{b}_n\sum_{m=0}^{\infty}\int_{\gamma}\sigma^{-n}\left(\frac{\varsigma^m}{\sigma^{m+1}}\right)d\sigma=\frac{1}{2\pi}\sum_{n=1}^{\infty}\bar{b}_n\sum_{m=0}^{\infty}\varsigma^m\int_0^{2\pi}e^{-i(n+m+1)\theta}\frac{d}{d\theta}\left(e^{i\theta}\right)d\theta=$$

$$=\frac{i}{2\pi}\sum_{n=1}^{\infty}\bar{b}_n\sum_{m=0}^{\infty}\varsigma^m\int_0^{2\pi}e^{-i(n+m)\theta}d\theta=\frac{1}{2\pi}\sum_{n=1}^{\infty}\bar{b}_n\sum_{m=0}^{\infty}\varsigma^m\left\{\int_0^{2\pi}\cos[(n+m)\theta]d\theta-i\int_0^{2\pi}\sin[(n+m)\theta]d\theta\right\}=0. \tag{47}$$

Upon inserting (46)–(47) in (45), it is seen that:

$$f(\varsigma)=i\sum_{n=0}^{\infty}b_n\varsigma^n\,. \tag{48}$$

Now we are going to determine the twisting rigidity $D$ indicated by formula (12), where we put the coefficients $a_n$.

At first we must transform polar moment of inertia $J_o$ {see formula (30)}, which we can write in the following equivalent form:

$$J_o=-\frac{1}{4}i\int_{\gamma}\left[w(\sigma)\bar{w}\left(\frac{1}{\sigma}\right)\right]\bar{w}\left(\frac{1}{\sigma}\right)dw(\sigma). \tag{49}$$

Infinite series (41) enables to show the following expressions involving the coefficients $a_n$:

$$\bar{w}\left(\frac{1}{\sigma}\right)=\sum_{n=0}^{\infty}\bar{a}_n\sigma^{-n}\,,$$

$$dw(\sigma)=\frac{d}{d\theta}\sum_{m=0}^{\infty}a_me^{im\theta}d\theta=\sum_{m=0}^{\infty}a_m im\left(e^{i\theta}\right)^m d\theta=i\sum_{m=1}^{\infty}a_m m\sigma^m d\theta\,. \tag{50}$$

We insert expressions (50) into (49), hence:

$$\bar{w}\left(\frac{1}{\sigma}\right)dw(\sigma)=i\left(\sum_{n=0}^{\infty}\bar{a}_n\sigma^{-n}\right)\left(\sum_{m=1}^{\infty}ma_m\sigma^m\right)d\theta\,. \tag{51}$$

Multiplying two infinite series in formula (51) by means of Cauchy method, we obtain a sum of two following series:

$$\bar{w}\left(\frac{1}{\sigma}\right)dw(\sigma)=i\sum_{n=0}^{\infty}c_n\sigma^n d\theta+i\sum_{n=1}^{\infty}c_{-n}\sigma^{-n}d\theta\,, \tag{52}$$

whereas

$$c_n = \sum_{r=0}^{\infty}(n+r)a_{n+r}\bar{a}_r\ , \quad c_{-n} = \sum_{r=0}^{\infty} r\bar{a}_{n+r}a_r \tag{53}$$

for $n$ = 0, 1, 2, .... .

To obtain polar moment of inertia of bone cross section in terms of coefficients $a_n$, two expressions (52) and (43) are substituted into (49):

$$J_0 = -\frac{1}{4}i\int_\gamma\left[w(\sigma)\overline{w}\left(\frac{1}{\sigma}\right)\right]\overline{w}\left(\frac{1}{\sigma}\right)dw(\sigma) = -\frac{i^2}{4}\int_0^{2\pi}\left(\sum_{n=0}^{\infty}b_n\sigma^n + \sum_{n=1}^{\infty}\bar{b}_n\sigma^{-n}\right)\left(\sum_{n=0}^{\infty}c_n\sigma^n + \sum_{n=1}^{\infty}c_{-n}\sigma^{-n}\right)d\theta =$$
$$= \frac{1}{4}\int_0^{2\pi}\left(b_0 + \sum_{n=1}^{\infty}b_n\sigma^n + \sum_{n=1}^{\infty}\bar{b}_n\sigma^{-n}\right)\left(c_0 + \sum_{n=1}^{\infty}c_n\sigma^n + \sum_{n=1}^{\infty}c_{-n}\sigma^{-n}\right)d\theta = \frac{1}{4}\left(B_0 + B_1 + B_2 + B_3 + B_4\right)\ , \tag{54}$$

where

$$B_0 \equiv \int_0^{2\pi}\left[c_0b_0 + \sum_{n=1}^{\infty}(b_0c_n + b_nc_0)\sigma^n + \sum_{n=1}^{\infty}(b_0c_{-n} + \bar{b}_nc_0)\sigma^{-n}\right] =$$
$$= 2\pi c_0b_0 + \sum_{n=1}^{\infty}\left[(c_nb_0 + c_0b_n)\int_0^{2\pi}e^{in\theta}d\theta\right] + \sum_{n=1}^{\infty}\left[(c_{-n}b_0 + c_0\bar{b}_n)\int_0^{2\pi}e^{-in\theta}d\theta\right] = 2\pi c_0b_0\ , \tag{55}$$

$$B_1 \equiv \int_0^{2\pi}\left(\sum_{n=1}^{\infty}b_n\sigma^n\right)\left(\sum_{n=1}^{\infty}c_{-n}\sigma^{-n}\right)d\theta = \int_0^{2\pi}\left[\sum_{n=1}^{\infty}b_nc_{-n} + \sum_{n=1}^{\infty}f_1(b_n, c_{-n})\sigma^n + \sum_{n=1}^{\infty}f_2(b_n, c_{-n})\sigma^{-n}\right]d\theta = 2\pi\sum_{n=1}^{\infty}b_nc_{-n}, \tag{56}$$

$$B_2 \equiv \int_0^{2\pi}\left(\sum_{n=1}^{\infty}\bar{b}_n\sigma^{-n}\right)\left(\sum_{n=1}^{\infty}c_n\sigma^n\right)d\theta = \int_0^{2\pi}\left[\sum_{n=1}^{\infty}\bar{b}_nc_n + \sum_{n=1}^{\infty}f_3(\bar{b}_n, c_n)\sigma^n + \sum_{n=1}^{\infty}f_4(\bar{b}_n, c_n)\sigma^{-n}\right]d\theta = 2\pi\sum_{n=1}^{\infty}b_nc_{-n}\ , \tag{57}$$

$$B_3 = \int_0^{2\pi}\left(\sum_{n=1}^{\infty}b_n\sigma^n\right)\left(\sum_{n=1}^{\infty}c_n\sigma^n\right)d\theta = 0, \tag{58}$$

$$B_4 = \int_0^{2\pi}\left(\sum_{n=1}^{\infty}\bar{b}_n\sigma^{-n}\right)\left(\sum_{n=1}^{\infty}c_{-n}\sigma^{-n}\right)d\theta = 0, \tag{59}$$

where $f_1$, $f_2$, $f_3$, $f_4$ are functions of constant coefficients: $b_n$, $\bar{b}_n$, $c_n$, $c_{-n}$.

In above formulae zero values of the following integrals are used:

$$\int_0^{2\pi} f_k\left(b_n, \bar{b}_n, c_n, c_{-n}\right)\cdot\sigma^{\pm n}d\theta = \int_0^{2\pi} f_k\left(b_n, \bar{b}_n, c_n, c_{-n}\right)\cdot e^{\pm in\theta}d\theta = 0\ , \tag{60}$$

for $n \neq 0$, $k$ = 1, 2, 3, 4.

We substitute (55)-(59) into formula (54), hence polar moment of inertia for bone cross section has the following form:

$$J_0 = \frac{\pi}{2}\left[b_0c_0 + \sum_{n=1}^{\infty}\left(b_nc_{-n} + \bar{b}_nc_n\right)\right]. \tag{61}$$

Now we must transform expression $D_0$ {see equation (20)} in terms of coefficients $a_n$.

At first we calculate two following expressions:

$$B_5 \equiv d\left[w(\sigma)\overline{w}(\bar{\sigma})\right], \tag{62}$$

$$B_6 \equiv f(\sigma) + \bar{f}(\bar{\sigma}). \tag{63}$$

We put formula (43) into (62):

$$B_5 = \frac{d}{d\sigma}\left[\sum_{n=0}^{\infty} b_n \sigma^n + \sum_{n=1}^{\infty} \bar{b}_n \sigma^{-n}\right]\frac{d\sigma}{d\theta}d\theta = \left[\sum_{n=1}^{\infty} nb_n\sigma^{n-1} - \sum_{n=1}^{\infty} n\bar{b}_n\sigma^{-n-1}\right]ie^{i\theta}d\theta = \\ = i\left[\sum_{n=1}^{\infty} nb_n\sigma^n - \sum_{n=1}^{\infty} n\bar{b}_n\sigma^{-n}\right]d\theta. \tag{64}$$

We put now formula (48) for $\varsigma = \sigma$ into (63), thus we obtain:

$$B_6 = i\sum_{m=0}^{\infty} b_m\sigma^m - i\sum_{m=0}^{\infty} \bar{b}_m\sigma^{-m} = i\left[\left(b_0 - \bar{b}_0\right) + \sum_{n=1}^{\infty} b_n\sigma^n - \sum_{n=1}^{\infty} \bar{b}_n\sigma^{-n}\right]. \tag{65}$$

From relations (64), (65) and taking into account contour integral (20), we determine expression $D_0$ in the following form:

$$D_0 = -\frac{i^2}{4}\int_0^{2\pi}\left[\left(b_0 - b_m\right) + \sum_{n=1}^{\infty} b_n\sigma^n - \sum_{n=1}^{\infty} \bar{b}_n\sigma^{-n}\right]\left[\sum_{n=1}^{\infty} nb_n\sigma^n - \sum_{n=1}^{\infty} n\bar{b}_n\sigma^{-n}\right]d\theta = \\ = \frac{1}{4}\left(B_7 + B_8 + B_9 + B_{10} + B_{11}\right), \tag{66}$$

where

$$B_7 \equiv \left(b_0 - \bar{b}_0\right)\sum_{n=1}^{\infty} n\left(b_n\int_0^{2\pi}\sigma^n d\theta - \bar{b}_n\int_0^{2\pi}\sigma^{-n}d\theta\right) = 0, \tag{67}$$

$$B_8 \equiv \int_0^{2\pi}\left(\sum_{n=1}^{\infty} b_n\sigma^n\right)\left(\sum_{n=1}^{\infty} nb_n\sigma^n\right)d\theta = 0, \tag{68}$$

$$B_9 \equiv -\int_0^{2\pi}\left(\sum_{n=1}^{\infty} b_n\sigma^n\right)\left(\sum_{n=1}^{\infty} n\bar{b}_n\sigma^{-n}\right)d\theta = -\int_0^{2\pi}\left[\sum_{n=1}^{\infty} nb_n\bar{b}_n + \sum_{n=1}^{\infty} f_5\left(b_n,\bar{b}_n\right)\sigma^n + \sum_{n=1}^{\infty} f_5\left(b_n,\bar{b}_n\right)\sigma^n\right]d\theta = -2\pi\sum_{n=1}^{\infty} nb_n\bar{b}_n, \tag{69}$$

$$B_{10} \equiv -\int_0^{2\pi}\left(\sum_{n=1}^{\infty} \bar{b}_n\sigma^{-n}\right)\left(\sum_{n=1}^{\infty} nb_n\sigma^n\right)d\theta = -\int_0^{2\pi}\left[\sum_{n=1}^{\infty} nb_n\bar{b}_n + \sum_{n=1}^{\infty} f_7\left(b_n,\bar{b}_n\right)\sigma^n + \sum_{n=1}^{\infty} f_8\left(b_n,\bar{b}_n\right)\sigma^n\right]d\theta = -2\pi\sum_{n=1}^{\infty} nb_n\bar{b}_n, \tag{70}$$

$$B_{11} \equiv \int_0^{2\pi}\left(\sum_{n=1}^{\infty} \bar{b}_n\sigma^{-n}\right)\left(\sum_{n=1}^{\infty} n\bar{b}_n\sigma^{-n}\right)d\theta = 0, \tag{71}$$

In above formulae zero values of the following integrals are used:

$$\int_0^{2\pi} f_k\left(b_n,\bar{b}_n\right)\cdot\sigma^{\pm n}d\theta = \int_0^{2\pi} f_k\left(b_n,\bar{b}_n\right)\cdot e^{\pm in\theta}d\theta = 0, \tag{71*}$$

for $n \neq 0$, $k$ = 5, 6, 7, 8.

If we put expressions (67)-(71) into formula (66), we obtain:

$$D_0 = -\pi\sum_{n=1}^{\infty} nb_n\bar{b}_n . \tag{72}$$

The resultant twisting moment (11)-(12) by virtue of results (61) and (72) has the following form:

$$M = \alpha G\left(J_0 + D_0\right) = \frac{\pi}{2}\alpha G\left[b_0c_0 + \sum_{n=1}^{\infty}(b_nc_{-n} + \bar{b}_nc_n - 2nb_n\bar{b}_n)\right]. \tag{73}$$

Combining the expressions (44) for $b_n$ and (53) for $c_n$ in formula (73), we obtain:

$$M = \frac{\pi}{2}\alpha G\left\{\left(\sum_{r=0}^{\infty} a_r \bar{a}_r\right)\left(\sum_{r=0}^{\infty} r a_r \bar{a}_r\right) + \sum_{n=1}^{\infty}\left[\left(\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right)\left(\sum_{r=0}^{\infty} r\bar{a}_{n+r} a_r\right) + \right.\right.$$
$$\left.\left. + \left(\sum_{r=0}^{\infty} \bar{a}_{n+r} a_r\right)\left(\sum_{r=0}^{\infty} (n+r) a_{n+r}\bar{a}_r\right) - 2n\left(\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right)\left(\sum_{r=0}^{\infty} \bar{a}_{n+r} a_r\right)\right]\right\}. \quad (74)$$

After reductions and simplifications formula (74) has the following form:

$$M = \frac{\pi}{2}\alpha G\left\{\left(\sum_{r=0}^{\infty} a_r \bar{a}_r\right)\left(\sum_{r=1}^{\infty} r a_r \bar{a}_r\right) + \sum_{n=1}^{\infty}\left[\left(\sum_{r=0}^{\infty} \bar{a}_{n+r} a_r\right)\left(\sum_{r=1}^{\infty} r a_{n+r}\bar{a}_r\right) + \right.\right.$$
$$\left.\left. + \left(\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right)\left(\sum_{r=1}^{\infty} r\bar{a}_{n+r} a_r\right) - n\left(\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right)\left(\sum_{r=0}^{\infty} \bar{a}_{n+r} a_r\right)\right]\right\}. \quad (75)$$

Taking into account the good known laws of conjugate complex numbers (for example, $z\bar{z} = |z|^2$ or $z + \bar{z} = 2\Re e(z)$), the resultant twisting moment (75) has the following form:

$$M = \frac{\pi}{2}\alpha G\left\{\left(\sum_{r=0}^{\infty} a_r \bar{a}_r\right)\left(\sum_{r=1}^{\infty} r a_r \bar{a}_r\right) + \sum_{n=1}^{\infty}\left[2\Re e\left(\sum_{r=0}^{\infty} \bar{a}_{n+r} a_r\right)\left(\sum_{r=1}^{\infty} r a_{n+r}\bar{a}_r\right) - n\left|\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right|^2\right]\right\}. \quad (76)$$

If all coefficients $a_n$ have real values, i.e. $a_n = \bar{a}_n$, then the resultant twisting moment in bone cross section has the following form:

$$M = \frac{\pi}{2}\alpha G\left\{\left(\sum_{r=0}^{\infty} a_r^2\right)\left(\sum_{r=1}^{\infty} r a_r^2\right) + \sum_{n=1}^{\infty}\left[\left[2\left(\sum_{r=0}^{\infty} a_{n+r} a_r\right)\left(\sum_{r=1}^{\infty} r a_{n+r} a_r\right) - n\left(\sum_{r=0}^{\infty} a_{n+r} a_r\right)^2\right]\right]\right\}. \quad (77)$$

Now we determine stresses $\tau_{zx}$, $\tau_{zy}$ {see formula (40)} in the terms of coefficients $a_n$, where functions $f(\varsigma)$ {see (48), (44)} and formula $w(\varsigma)$ {see (41)} are used:

$$\tau_{zx} - i\tau_{zy} = G\alpha\left\{\frac{\frac{d}{d\varsigma}\left[i\sum_{n=0}^{\infty}\left(\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right)\varsigma^n\right]}{\frac{d}{d\varsigma}\left(\sum_{n=0}^{\infty} a_n\varsigma^n\right)} - i\sum_{n=0}^{\infty}\bar{a}_n\bar{\varsigma}_n\right\}. \quad (78)$$

After differentiation formula (78) has the following form:

$$\tau_{zx} - i\tau_{zy} = iG\alpha\left\{\frac{\sum_{n=1}^{\infty}\left(\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r\right)n\varsigma^{n-1}}{\sum_{n=1}^{\infty} n a_n\varsigma^{n-1}} - \sum_{n=0}^{\infty}\bar{a}_n\bar{\varsigma}^n\right\}. \quad (79)$$

## 4. Applications

*Example*

As an illustration of the foregoing procedure we consider a cardioid bone cross section in $(x,y)$-plane and polar coordinates $(r,\beta)$ in the following form:

$$r = 2c(1+\cos\beta), \tag{80}$$

where $r=\sqrt{x^2+y^2}$, $\cos\beta=\dfrac{x}{r}$, $0\le\beta<2\pi$, $c>0$ is the arbitrary positive constant.

Now we must show that a suitable function which maps cardioid from $(x,y)$ plane into the unit circle in $(\xi,\eta)$ plane has the following form:

$$z=c(1-\varsigma)^2,\ \ \varsigma=\xi+i\eta,\ \ z=x+iy. \tag{81}$$

Fig. 3 shows conformal mapping of the cardioid bone cross section $R$ from the complex $z=(x,y)$ plane onto the unit circle in the complex $\varsigma=(\xi,\eta)$ plane bounded by the round $\gamma$.

*Solution*

By virtue of equations (80), (81) the following expressions of modulus of complex number are true:

$$|z|=c|1-\xi-i\eta|^2,\ \ |z|=r=\sqrt{x^2+y^2}. \tag{82}$$

Combining expressions (82), we obtain:

$$r=\sqrt{x^2+y^2}=c|1-\xi-i\eta|^2=\left[(1-\varsigma)^2+\eta^2\right]. \tag{83}$$

From formula (81) it follows that:

$$x+iy=c(1-\xi-i\eta)^2. \tag{84}$$

Equating the coefficients of the real and imaginary parts in formula (84), we obtain:

$$x=c[(1-\xi)^2-\eta^2],\ \ y=-2c\eta(1-\xi). \tag{85}$$

Hence, it follows from (80) that:

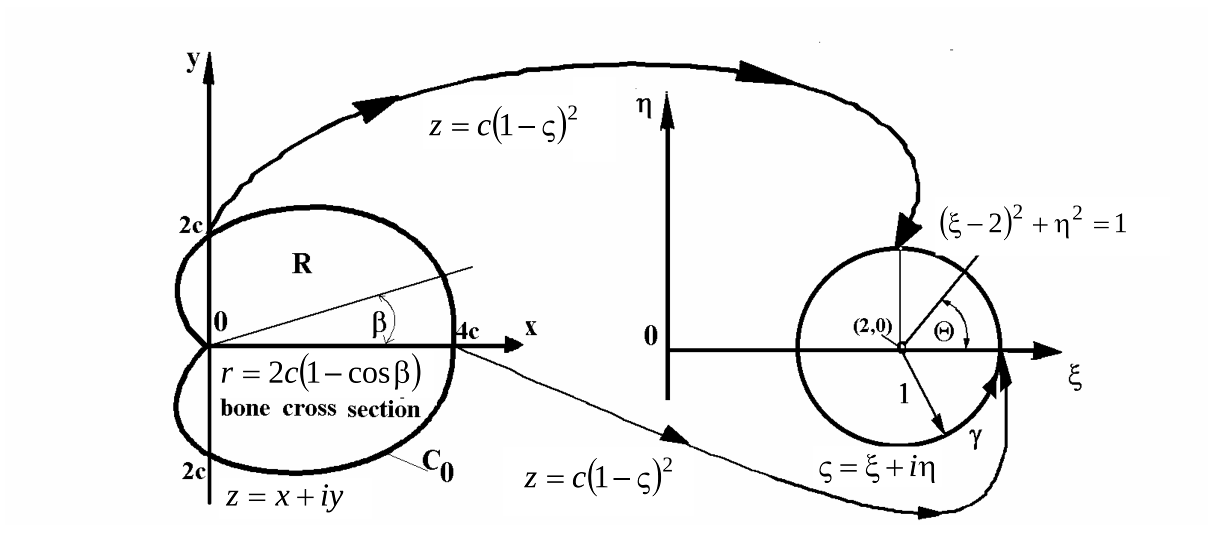


Fig. 3. Conformal mapping: $z=c(1-\varsigma)^2$ of the cardioid: $r=2c(1-\cos\beta)$ of bone cross section $R$ in $z=x+iy$ plane onto the unit circle: $(\xi-2)^2+\eta^2=1$ in $\varsigma=\xi+i\eta$ plane bounded by the round $\gamma$.

$$\sqrt{x^2+y^2}=2c\left(1+\frac{x}{\sqrt{x^2+y^2}}\right). \tag{86}$$

If we put formulae (83), (85) in the right and left members of equation (86), then we obtain the unit circle in the $(\xi,\eta)$-plane, whose center lays in the point $(2,0)$, therefore:

$$(\xi-2)^2+\eta^2=1. \tag{87}$$

This fact proofs that function (81) maps cardioid onto the unit circle. Mapping function (81) may be written in the form of multinomial:

$$z=c-2c\varsigma+c\varsigma^2, \tag{88}$$

where only non-vanishing coefficients $a_n$ are as follows:

$$a_0=c,\ a_1=-2c,\ a_2=c. \tag{89}$$

If we insert coefficients presented in (89) into (77), then twisting moment in the cardioid bone cross section has the following form:

$$M=\frac{\pi}{2}\alpha G\left(36c^4-18c^4+16c^4\right)=17\pi\alpha Gc^4. \tag{90}$$

Now we put coefficients presented in expression (89) into formula (79) to find the tangential stresses in the cardioid bone cross section:

$$\tau_{zx}-i\tau_{zy}=iG\alpha\left\{\frac{-4c^2+2c^2\varsigma}{-2c+2c\varsigma}-c+2c\bar{\varsigma}-c\bar{\varsigma}^2\right\}. \tag{91}$$

After simplifications formula (91) has the following form:

$$\tau_{zx}-i\tau_{zy}=iG\alpha c\left\{\frac{\varsigma-2}{\xi-1}-\overline{(1-\varsigma)^2}\right\}. \tag{92}$$

In the right hand side of equation (92) we eliminate variable $\varsigma$ through the complex number $z$ by means of formula (81), hence after simplifications we obtain:

$$\tau_{zx}-i\tau_{zy}=iG\alpha c\left\{\frac{\sqrt{c}+\sqrt{z}}{\sqrt{z}}-\frac{\bar{z}}{c}\right\}. \tag{93}$$

Complex number in trigonometric form: $z=r(\cos\beta+i\sin\beta)$ is put in the right hand side of equation (93), thus after calculations we have:

$$\tau_{zx}-i\tau_{zy}=iG\alpha c\left[\sqrt{\frac{c}{r}}\left(\cos\frac{\beta}{2}-i\sin\frac{\beta}{2}\right)+1-\frac{r}{c}\left(\cos\beta-i\sin\beta\right)\right]. \tag{94}$$

We put cardioid function (81) in the left hand side of equation (94) and we equate coefficients of the real and imaginary parts of the left and right hand sides of equation (94). Hence we obtain shear stresses on the boundary of cardioid bone cross section in the following form (for $0\le\beta<2\pi$):

$$\tau_{zx}^{(b)}=G\alpha c\left(\frac{1}{2}\tan\frac{\beta}{2}-8\sin\frac{\beta}{2}\cos^3\frac{\beta}{2}\right),$$

$$\tau_{zy}^{(b)}=G\alpha c\left(-\frac{3}{2}+4\cos^2\frac{\beta}{2}\cos\beta\right). \tag{95}$$

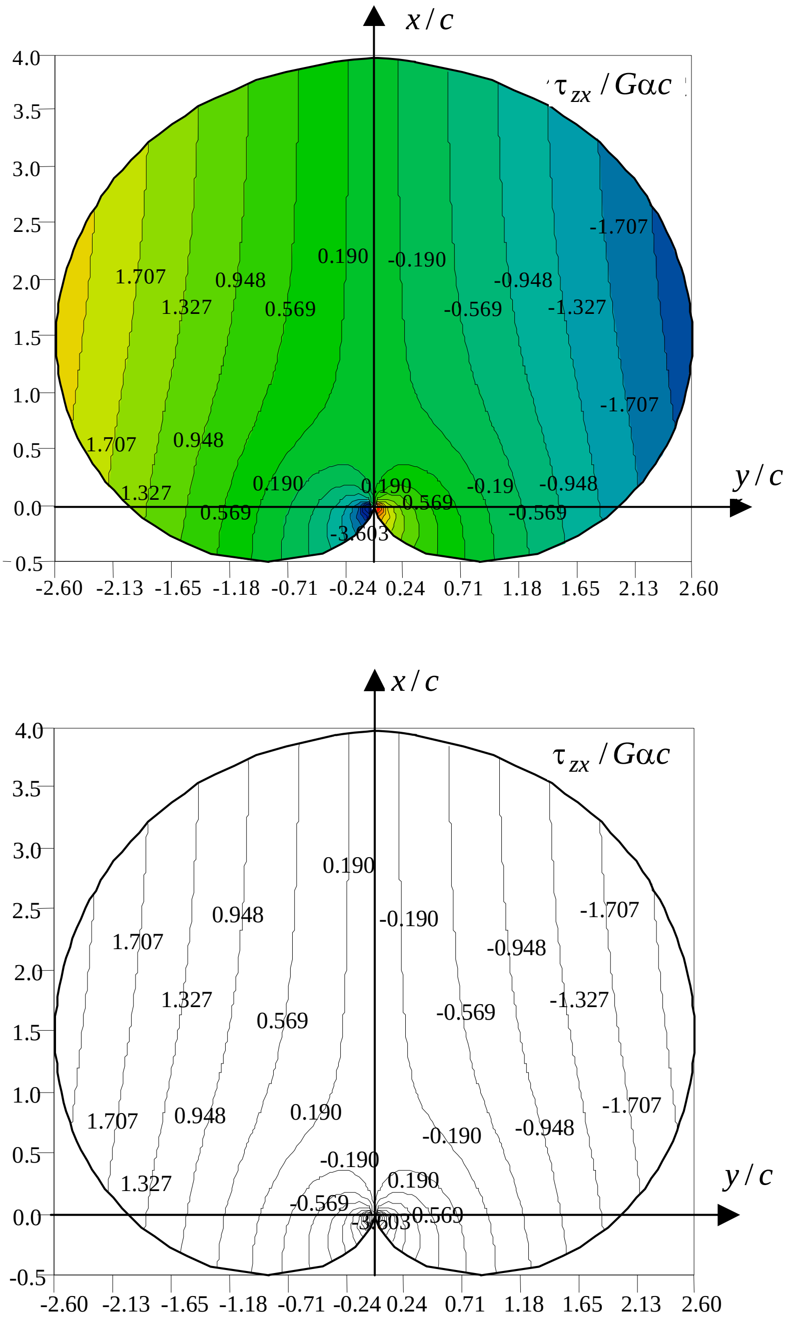


Fig. 4. Shear stress $\tau_{zx}$ distribution on the boundary line of the cardioid bone cross section.

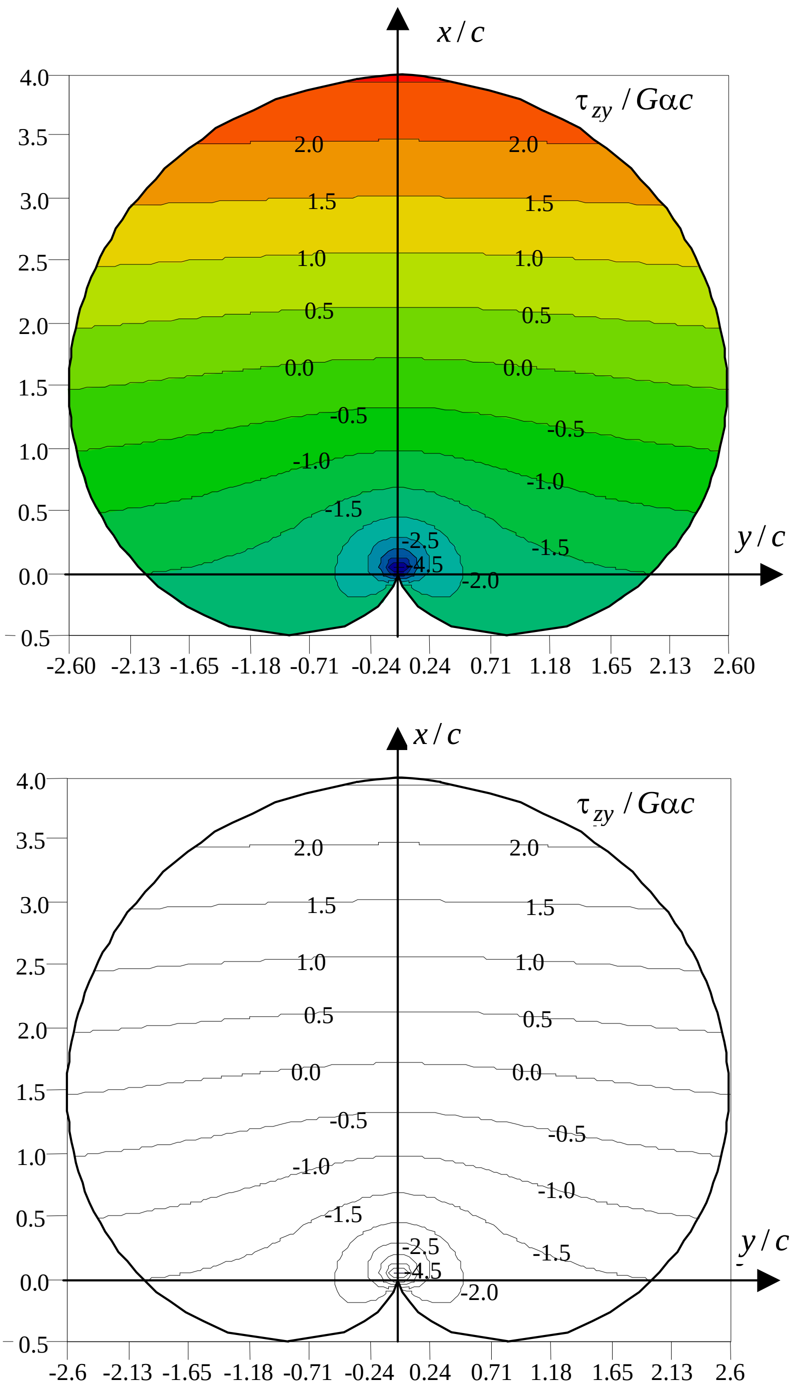


Fig. 5. Shear stress $\tau_{zy}$ distribution on the boundary line of the cardioid bone cross section.

## 5. Numerical example

By virtue of (94) we perform now numerical calculations of the shear stresses on the cardioid bone cross section. The results are given in Fig. 4 and Fig. 5 in polar co-ordinates with the pole $0$.

We obtain negative and positive values of the shear stresses. The maximal values of shear stresses $\tau_{zx}$ are situated in neighbourhood of the origin $0$ of polar co-ordinates, i.e. in the point near to the boundary of cardioid where boundary line is not differentiable but continuous (see Fig. 4).

To obtain real values of shear stresses, we must multiply the dimensionless values of shear stresses indicated in Fig. 4 and Fig. 5 by the dimensional factor $G\alpha c$ in N/m$^2$, where $G$ is the shear modulus of bone material in Pa, $c$ is the positive coefficient of scale indicated in bone cardioid cross section in m (Fig. 4, Fig. 5), $\alpha$ is the twist per unit bone length in 1/m.

The maximal difference values of changes of shear stresses $\tau_{zy}$ are situated near to the pole origin $0$ of polar co-ordinates, i.e. in the point in boundary of cardioid where boundary line is not differentiable but continuous (see Fig. 5).

## 6. Conclusions

To accurately calculate bio-bearing (joint) parameters we must know shear stresses both in bone cross section and lateral surfaces, which influence on the human joint performance. Therefore in this paper, we determined the resultant magnitude of shear stress vector in human joint produced by the continuous load and by moment of twisting in the human bone.

In calculations, we were taking into account that real human bones had non-circular cross sections. For example, we took into account the cardioid bone cross section.

The maximal values of the shear stresses caused by the twisting moment occurred in the boundary of bone cross sections even if the boundary had irregular shapes.

In particular cases, the all obtained results tended to the simple particular solutions, which are good known in classical twisting theory.

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# РЕШЕНИЕ С ПОМОЩЬЮ КОНФОРМНОГО ОТОБРАЖЕНИЯ ЗАДАЧИ О КРУЧЕНИИ КОСТИ ЧЕЛОВЕКА

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Задача кручения кости человека при упругом поведении костной ткани может быть решена с помощью решения задачи Неймана или Дирихле, определяя функцию, которая является гармонической в области поперечного сечения кости и удовлетворяет заданным граничным условиям на границе поперечного сечения. Однако практическое определение этой функции представляет собой очень сложную задачу. Поэтому в данной работе используется иной метод решения задачи, применяя конформное отображение для исходной области нерегулярной, некруговой формы. В результате отображения область поперечного сечения отображается на круг единичного радиуса. Получены аналитические формулы для вычисления распределения касательных напряжений и результирующего момента при кручении кости с произвольной формой поперечного сечения. В качестве примера рассмотрено кручение кости с поперечным сечением, ограниченным контуром в форме кардиоиды. В частности, определены области максимальных напряжений. Библ. 6.

Ключевые слова: кручение вала, аналитическое решение, касательные напряжения, момент кручения, конформное отображение, поперечное сечение произвольной формы

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