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HEMODYNAMICS OF THE HUMAN CARDIOVASCULAR SYSTEM IN TURBULENT BLOOD FLOW

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Abstract: The formulas for the determination of differential operators in the subprojective space referred to the nonholonomic frame are obtained. These formulas may be helpful for the description of the turbulent blood flow.

Key words: hemodynamics, subprojective space, geodesic lines, differential operators

Introduction

The given paper is devoted to modelling of the turbulent blood flow. In it, some problems of the geometric theory of the stationary blood flow are considered. Below all human cardiovascular system [1] is represented as a subprojective space [2] and the vessels are represented as geodesic lines of the three-dimensional subprojective space. Such vessels we have named "geodesic" [3].

We understand as the three-dimensional subprojective space an affinely connected space, whose geodesic lines in some system of coordinates can be presented by a system of two equations, from which one equation is linear. If the subprojective space is mapped on the Euclidean one, images of the geodesic lines of this space will be the lines placed in the two-dimensional planes E^2 of the three-dimensional Euclidean space. Then it is possible to name a pre-image of the plane E^2 as the “plane” in the subprojective space, that is a two-dimensional surface expressed in the given frame by a simple equation. Moreover in the subprojective space the geodesic lines lay in the two-dimensional "planes" going through a generic point.

Poiseuille noticed that at large fluid velocities the relationship between the pressure and the flow was not linear any more. Reynolds studying flow in long cylindrical pipe, in which he made an injection of a paint in an axial stream, showed that the fluid flow remained laminar when velocity was less than some critical one, after reaching critical velocity the turbulent flow occurred. He obtained the following formula:

$$\mathrm{Re} = \frac{\nu D \rho}{\eta},$$

where v is the average velocity, D is the pipe diameter, ρ is the density, η is the viscosity, Re is the Reynolds number [4].

If the Reynolds number exceeds 2000 turbulence begins to appear.

In all minor vessels, which are "the channels of resistance", the blood flow has a laminar character. However the normal turbulence is present in the ventricles and auricles; it is very favourable as blood mixing in the heart takes place. The turbulence can also take place in the aorta. Here the pulsing flow promotes the turbulence even when Reynolds number is much less than 2000 for the translational movement of blood. However the intermittence of the blood flow hinders the formation of the turbulence, the development of which takes some time.

The pathological modifications in the form of badges promotes the local development of a turbulence right outside the site where the diameter of a vessel is reduced.

The turbulent flow is more complicated than laminar and to describe it more complicated mathematical model is required.

The nonholonomic frame of the subprojective space and the differential operators

Let blood in a vessel flows with turbulences, that is we can accept that from one point to another a particle of blood spins along some path. Thus we assume that the blood vessel is in the three-dimensional subprojective space.

The set of all the paths along which blood moves from one point to another in the three-dimensional subprojective space by analogy with [5] we will name a nonholonomic manifold. The three-dimensional subprojective space V^3 can be considered as a variety referred to the nonholonomic frame.

In the tangential space to the three-dimensional subprojective space we shall set a frame defined by the point $x \in V^3$ and the vectors $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$. The equations of transition of such a frame look like:

$$dx = \omega^i \mathbf{e}_i ,\; de_i = \omega_i^j \mathbf{e}_j + \omega^j \mathbf{e}_{ij} , \tag{1}$$

where $\mathbf{e}_{ij}$ are vectors forming with $\mathbf{e}_i$ a frame of the second order. The indexes *i, j, k* here and further accept the values 1, 2, 3. For the given subprojective space V^3 $\mathbf{e}_{ij} \neq \mathbf{e}_{ji}$ [6]. We shall name such a frame nonholonomic, and the subprojective space as referred total nonholonomic frame. In [7] the blood flow in the three-dimensional holonomic subprojective space is considered.

The forms ω^1, ω^2, ω^3 are linearly independent and $\omega^1 \wedge \omega^2 \wedge \omega^3 \neq 0$.

The differential forms ω^i and ω^j from the equations (1) satisfy the equations of the structure of the subprojective space:

$$D\omega^i = \omega^j \wedge \omega_j^i ,\; D\omega_j^i = \omega_j^k \wedge \omega_k^i + R_{jkl}^i \,\omega^k \wedge \omega^1 \tag{2}$$

where R_{jkl}^i is tensor of a curvature of the subprojective space.

Whereas the subprojective space is a special case of the Riemannian space, as a structural group of this space we shall take an orthogonal group *O*(n), whose invariant forms satisfy the equations $\sigma_i^j + \sigma_j^i = 0$, where $\omega_j^i\,(\omega^i = 0) = \sigma_j^i$. Then the forms ω_j^i also satisfy the equations [8]:

$$\omega_j^i + \omega_i^j = 0 ,\; \omega_i^i = 0 . \tag{3}$$

As well as in [7] let us find an expression for the gradient of an arbitrary function φ, the divergence of a vector of the blood velocity and also the curl of a vector of the velocity in the given subprojective space. An expression for the gradient as in [9] is:

$$\text{grad } \varphi = \frac{e^1 d\phi \wedge \omega^2 \wedge \omega^3 + e^2 d\phi \wedge \omega^3 \wedge \omega^1 + e^3 d\phi \wedge \omega^1 \wedge \omega^2}{\omega^1 \wedge \omega^2 \wedge \omega^3} , \tag{4}$$

where e^1, e^2, e^3 are mutual vectors to the vectors of the given frame.

Let $\mathbf{v}$ be a vector of the velocity of a blood particle, which we shall present as $\mathbf{v} = v^i \mathbf{e_i}$. Differentiating this equation and using the second equation from (1) we obtain:

$$d\mathbf{v} = \left(dv^i + v^j \omega_j^i\right)\mathbf{e}_i + v^i \omega^j \mathbf{e}_{ij} . \tag{5}$$

As a frame, in the tangential space to the subprojective space V^3, we shall consider an orthonormalized frame. Then $\mathbf{e_i}^2 = 1$. Differentiating the latter equation, taking into account (1) we shall obtain:

$$\omega_i^i + \omega^j \mathbf{e}_i\, \mathbf{e}_{ij} = 0 .$$

Taking into account (3) and due to the linear independence of the forms ω^j the latter equation has the following form:

$$\mathbf{e_i}\, \mathbf{e}_{ij} = 0. \qquad (6)$$

From the equalities (6) we obtain:

$$\begin{array}{lll} \mathbf{e_1 e_{12}} = 0, & \mathbf{e_2 e_{21}} = 0, & \mathbf{e_3 e_{31}} = 0, \\ \mathbf{e_1 e_{13}} = 0, & \mathbf{e_2 e_{22}} = 0, & \mathbf{e_3 e_{32}} = 0, \\ \mathbf{e_1 e_{11}} = 0, & \mathbf{e_2 e_{23}} = 0, & \mathbf{e_3 e_{33}} = 0. \end{array} \qquad (7)$$

Because of the equalities (7), we shall have:

$$\mathbf{e}_{ij} = a_{ij}^k \mathbf{e}_k \ \ (k \neq i). \qquad (8)$$

Let us designate through $d\tau$ an elementary volume in the blood. Then we shall determine a divergence of a vector of velocity **v** using the Gauss theorem for the volume of a parallelepiped formed in an arbitrary point of a vessel by the vectors of three arbitrary elementary transitions $d_1\mathbf{x}$, $d_2\mathbf{x}$, $d_3\mathbf{x}$. Then $d\tau = \omega^1 \wedge \omega^2 \wedge \omega^3\, \mathbf{e}_1\, \mathbf{e}_2\, \mathbf{e}_3$. After some simple calculations we conclude:

$$\text{div}\,\mathbf{v}\; d\tau = d_1\mathbf{v} d_2\mathbf{x} d_3\mathbf{x} + d_2\mathbf{v} d_3\mathbf{x} d_1\mathbf{x} + d_3\mathbf{v} d_1\mathbf{x} d_2\mathbf{x}. \qquad (9)$$

Using (8) the equation (5) takes the form:

$$d\mathbf{v} = \left(dv^i + v^j \omega_j^i\right)\mathbf{e}_i + v^i \omega^j a_{ij}^k\, \mathbf{e}_k, \ (k \neq i).$$

Then the formula (9) will look like:

$$\omega^1 \wedge \omega^2 \wedge \omega^3 \ \text{div}\,\mathbf{v} = \begin{vmatrix} d_1 v^1 + v^j \overset{1}{\omega}{}^1_j + v^k a^1_{kj} \overset{1}{\omega}{}^j \ (k \neq 1) & \overset{2}{\omega}{}^1 & \overset{3}{\omega}{}^1 \\ d_1 v^2 + v^j \overset{1}{\omega}{}^2_j + v^k a^2_{kj} \overset{1}{\omega}{}^j \ (k \neq 2) & \overset{2}{\omega}{}^2 & \overset{3}{\omega}{}^2 \\ d_1 v^3 + v^j \overset{1}{\omega}{}^3_j + v^k a^3_{kj} \overset{1}{\omega}{}^j \ (k \neq 3) & \overset{2}{\omega}{}^3 & \overset{3}{\omega}{}^3 \end{vmatrix} +$$

$$+ \begin{vmatrix} d_2 v^1 + v^j \overset{2}{\omega}{}^1{}_j + v^k a^1{}_{kj} \overset{2}{\omega}{}^j \ (k \neq 1) & \overset{3}{\omega}{}^1 & \overset{1}{\omega}{}^1 \\ d_2 v^2 + v^j \overset{2}{\omega}{}^2{}_j + v^k a^2_{kj} \overset{2}{\omega}{}^j \ (k \neq 2) & \overset{3}{\omega}{}^2 & \overset{1}{\omega}{}^2 \\ d_2 v^3 + v^j \overset{2}{\omega}{}^3_j + v^k a^3_{kj} \overset{2}{\omega}{}^j \ (k \neq 3) & \overset{3}{\omega}{}^3 & \overset{1}{\omega}{}^3 \end{vmatrix} + \begin{vmatrix} d_3 v^1 + v^j \overset{3}{\omega}{}^1_j + v^k a^1_{kj} \overset{3}{\omega}{}^j \ (k \neq 1) & \overset{1}{\omega}{}^1 & \overset{2}{\omega}{}^1 \\ d_3 v^2 + v^j \overset{3}{\omega}{}^2_j + v^k a^2_{kj} \overset{3}{\omega}{}^j \ (k \neq 2) & \overset{1}{\omega}{}^2 & \overset{2}{\omega}{}^2 \\ d_3 v^3 + v^j \overset{3}{\omega}{}^3_j + v^k a^3_{kj} \overset{3}{\omega}{}^j \ (k \neq 3) & \overset{1}{\omega}{}^3 & \overset{2}{\omega}{}^3 \end{vmatrix}.$$

After the transformations in the right side we obtain:

$$\omega^1 \wedge \omega^2 \wedge \omega^3 \ \text{div}\,\mathbf{v} = \left(dv^1 + v^j \omega_j^1\right) \wedge \omega^2 \wedge \omega^3 + \left(dv^2 + v^j \omega_j^2\right) \wedge \omega^3 \wedge \omega^1 +$$

$$+ \left(dv^3 + v^j \omega_j^3\right) \wedge \omega^1 \wedge \omega^2 + (v^k \underset{k \neq i}{a}{}^{i}_{ki})\, \omega^1 \wedge \omega^2 \wedge \omega^3, \qquad (10)$$

where in the last term sum over i, and then sum over $k \neq i$ are supposed.

To determine a rotor of the vector of blood velocity we shall use the formula:

$$\iiint rot\,\mathbf{v} d\tau = -\oiint [\mathbf{v} d\boldsymbol{\sigma}],$$

where $d\boldsymbol{\sigma}$ is vector of the element of a surface.

Having applied this formula to the volume $d\tau = w^1 \wedge w^2 \wedge w^3\ \mathbf{e_1}\ \mathbf{e_2}\ \mathbf{e_3}$, we shall obtain:

$$- \mathrm{rot}\, \mathbf{v}\, d\tau = [(\mathbf{v} + d_1\mathbf{v}), (\mathbf{d}\sigma_{23} + d_1\, \mathbf{d}\sigma_{23})] + [\mathbf{v}, \mathbf{d}\sigma_{32}] + [(\mathbf{v} + d_2\mathbf{v}), (\mathbf{d}\sigma_{31} + d_2\, \mathbf{d}\sigma_{31})] +$$
$$+ [\mathbf{v}, \mathbf{d}\sigma_{13}] + [(\mathbf{v} + d_3\mathbf{v}), (\mathbf{d}\sigma_{12} + d_3\, \mathbf{d}\sigma_{12})] + [\mathbf{v}, \mathbf{d}\sigma_{21}],$$

where $\mathbf{d}\sigma_{ij} = [d_i\mathbf{x},\, d_j\mathbf{x}]$ is element of a surface in a point x formed by vectors $d_i\mathbf{x}$ and $d_j\mathbf{x}$.

As $d_1(\mathbf{d}\sigma_{23}) + d_2\, (\mathbf{d}\sigma_{31}) + d_3\, (\mathbf{d}\sigma_{12}) = 0$, we shall write:

$$- \mathrm{rot}\, \mathbf{v}\, d\tau = [d_1\, \mathbf{v}, [d_2\, \mathbf{x}, d_3\, \mathbf{x}]] + [d_2\, \mathbf{v}, [d_3\, \mathbf{x}, d_1\, \mathbf{x}]] + [d_3\, \mathbf{v}, [d_1\, \mathbf{x}, d_2\, \mathbf{x}]].$$

We shall rewrite the latter in a final form, making a series of transformations:

$$\mathrm{rot}\, \mathbf{v}\, d\tau = - \mathbf{e}_i\, \omega^i \wedge \omega^j \wedge \left(dv^k + v^1\omega_1^k\right)\left(\mathbf{e}_j\mathbf{e}_k\right) - \mathbf{e}_i\, \omega^i \wedge \omega^j \wedge (v^k\omega^1 \underset{s\neq k}{a^{\,l}_{kl}})\left(\mathbf{e}_j\mathbf{e}_s\right). \qquad (11)$$

In the orthonormalized frame the equation (11) will be noted as:

$$- \mathrm{rot}\, \mathbf{v}\, d\tau = \mathbf{e}_1\, (\omega^1 \wedge \omega^2 \wedge (dv^2 + v^k\, \omega^2{}_k) + \omega^1 \wedge \omega^3 \wedge (dv^3 + v^k\, \omega^3{}_k) + \omega^1 \wedge \omega^2 \wedge \omega^3\, (v^k \underset{2\neq k}{a^{\,2}_{k3}}) - \omega^1 \wedge \omega^2 \wedge \omega^3\, (v^k \underset{3\neq k}{a^{\,3}_{k2}})) + \mathbf{e_2}\, (\omega^2 \wedge \omega^1 \wedge (dv^1 + v^k\, \omega^1{}_k) + \omega^2 \wedge \omega^3 \wedge (dv^3 + v^k\, \omega^3{}_k) -$$
$$- \omega^1 \wedge \omega^2 \wedge \omega^3\, (v^k \underset{1\neq k}{a^{\,1}_{k3}}) + \omega^1 \wedge \omega^2 \wedge \omega^3\, (v^k \underset{3\neq k}{a^{\,3}_{k1}})) + \mathbf{e_3}\, (\omega^3 \wedge \omega^1 \wedge (dv^1 + v^k\, \omega^1{}_k) + \omega^3 \wedge$$
$$\wedge\, \omega^2 \wedge (dv^2 + v^k\, \omega^2{}_k) + \omega^1 \wedge \omega^2 \wedge \omega^3\, (v^k \underset{1\neq k}{a^{\,1}_{k2}}) - \omega^1 \wedge \omega^2 \wedge \omega^3\, (v^k \underset{2\neq k}{a^{\,2}_{k1}})).$$

For the orthogonal frame $d\tau = \omega^1 \wedge \omega^2 \wedge \omega^3$ the latter will be transformed as:

$$- \mathrm{rot}\, \mathbf{v}\, d\tau = \mathbf{e_1}\, (\omega^1 \wedge \omega^2 \wedge (dv^2 + v^k\, \omega^2{}_k) + \omega^1 \wedge \omega^3 \wedge (dv^3 + v^k\, \omega^3{}_k) + d\tau\, (v^k \underset{2\neq k}{a^{\,2}_{k3}} - v^k \underset{k\neq 3}{a^{\,3}_{k2}})) +$$
$$+ \mathbf{e_2}\, (\omega^2 \wedge \omega^1 \wedge (dv^1 + v^k\, \omega^1{}_k) + \omega^2 \wedge \omega^3 \wedge (dv^3 + v^k\, \omega^3{}_k) + d\tau\, (v^k \underset{3\neq k}{a^{\,3}_{k1}} - v^k \underset{1\neq k}{a^{\,1}_{k3}})) + \mathbf{e_3}\, (\omega^3 \wedge$$
$$\wedge\, \omega^1 \wedge (dv^1 + v^k\, \omega^1{}_k) + \omega^3 \wedge \omega^2 \wedge (dv^2 + v^k\, \omega^2{}_k) + d\tau\, (v^k \underset{1\neq k}{a^{\,1}_{k2}} - v^k \underset{2\neq k}{a^{\,2}_{k1}})). \qquad (12)$$

The expression for the curl of a vector of blood velocity is more complicated in this case than a similar expression in the Euclidean space [9] and in the holonomic subprojective space [7].

Some equations of the hemodynamics in the subprojective space

The formulas obtained for the gradient, divergence and curl allow to note the basic equations of hemodynamics when representing the human cardiovascular system as the subprojective space referred to a nonholonomic frame.

The equation $\frac{\partial \rho}{\partial t} = -\, \mathrm{div}\, (\rho\mathbf{v})$ is named as continuity equation. In view of the non-compressibility of blood its volumetric expenditure through a closed surface S should be equal to zero. The latter because of the Gauss theorem gives

$$\mathrm{div}\, \mathbf{v} = 0. \qquad (13)$$

Considering (13), the equation (10) will look like:

$$(dv^1 + v^j\, \omega^1{}_j) \wedge \omega^2 \wedge \omega^3 + (dv^2 + v^j\, \omega^2{}_j) \wedge \omega^3 \wedge \omega^1 +$$
$$+ (dv^3 + v^j\, \omega^3{}_j) \wedge \omega^1 \wedge \omega^2 + (v^k \underset{k\neq i}{a^{\,i}_{ki}})\, \omega^1 \wedge \omega^2 \wedge \omega^3 = 0. \qquad (14)$$

Selecting the vector $\mathbf{e_3}$ along the direction of the tangent of the blood streamline, we note $\mathbf{v} = v\, \mathbf{e_3}$. Then (14) may be rewritten in the following form:

$$v\omega^1{}_3 \wedge \omega^2 \wedge \omega^3 + v\omega^2{}_3 \wedge \omega^3 \wedge \omega^1 + (dv) \wedge \omega^1 \wedge \omega^2 + (v^k \underset{k\neq i}{a}{}^i_{ki}) \omega^1 \wedge \omega^2 \wedge \omega^3 = 0.$$

Having entered the notations $\omega^1{}_3 = - \omega^3{}_1 = q_i \omega^i = q$; $\omega^3{}_2 = - \omega^2{}_3 = p_i \omega^i = p$, the last equation will look like:

$$vq_1 \omega^1 \wedge \omega^2 \wedge \omega^3 - vp_2 \omega^2 \wedge \omega^3 \wedge \omega^1 + dv \wedge \omega^1 \wedge \omega^2 + v (a^1{}_{31} + a^2{}_{32}) \omega^1 \wedge \omega^2 \wedge \omega^3 = 0 \text{ or}$$

$$\left(\frac{d \ln v}{ds}\right) \omega^1 \wedge \omega^2 \wedge \omega^3 = (p_2 - q_1 - (a^1{}_{31} + a^2{}_{32})) \omega^1 \wedge \omega^2 \wedge \omega^3.$$

Finally we shall note:

$$\frac{d \ln v}{ds} = p_2 - q_1 - (a^1{}_{31} + a^2{}_{32}). \tag{15}$$

In the same way as it is done in [10] it is possible to show that the right side of the equation (15) is an average curvature of the field of vectors or an average curvature of the blood streamlines. Thus it is proved that the derivative of the logarithm of the blood velocity in the direction of the streamline is equal to an average curvature of congruences of blood streamlines at each point of the blood stream.

The right side of equation (15) will vanish only when $(p_2 - q_1)$ is equal to the sum of the first and second coordinates of vectors of the second order $\mathbf{e}_{31}$ and $\mathbf{e}_{32}$, respectively.

The congruence of the lines, for which $p_2 - q_1 - (a^1{}_{31} + a^2{}_{32}) = 0$, we shall name a minimum congruence in the given subprojective space.

Thus it is proved that the magnitude of the velocity of the stream of blood in nonholonomic subprojective space is fixed along a certain path in the only case when the given path represents a line belonging to the minimum congruence.

Let the rotational vector $\mathbf{V}$ of a vector of the blood velocity be

$$\mathbf{V} = 1/2 \text{ rot } \mathbf{v} = 1/2\, V^i \mathbf{e}_i. \tag{16}$$

Then from the formula (12) we shall define the components of the curl:

$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^1 =$$
$$= \omega^1 \wedge \omega^2 \wedge (dv^2 + v^k \omega^2{}_k) + \omega^1 \wedge \omega^3 \wedge (dv^3 + v^k \omega^3{}_k) + \omega^1 \wedge \omega^2 \wedge \omega^3 (v^k \underset{2\neq k}{a}{}^2_{k3} - v^k \underset{k\neq 3}{a}{}^k_{k2}),$$

$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^2 =$$
$$= \omega^2 \wedge \omega^1 \wedge (dv^1 + v^k \omega^1{}_k) + \omega^2 \wedge \omega^3 \wedge (dv^3 + v^k \omega^3{}_k) + \omega^1 \wedge \omega^2 \wedge \omega^3 (v^k \underset{3\neq k}{a}{}^3_{k1} - v^k \underset{1\neq k}{a}{}^1_{k3}),$$

$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^3 =$$
$$= \omega^3 \wedge \omega^1 \wedge (dv^1 + v^k \omega^1{}_k) + \omega^3 \wedge \omega^2 \wedge (dv^2 + v^k \omega^2{}_k) + \omega^1 \wedge \omega^2 \wedge \omega^3 (v^k \underset{1\neq k}{a}{}^1_{k2} - v^k \underset{k\neq 2}{a}{}^2_{k1}).$$

Taking into account that the vector $\mathbf{e}_3$ is directed on the tangent to the streamline, the last formulas will look like:

$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^1 = \omega^1 \wedge \omega^2 \wedge v\omega^2{}_3 + \omega^1 \wedge \omega^3 \wedge dv + \omega^1 \wedge \omega^2 \wedge \omega^3 (va^2{}_{33}),$$
$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^2 = \omega^2 \wedge \omega^1 \wedge v\omega^1{}_3 + \omega^2 \wedge \omega^3 \wedge dv + \omega^1 \wedge \omega^2 \wedge \omega^3 (-va^1{}_{33}),$$
$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^3 = \omega^3 \wedge \omega^1 \wedge v\omega^1{}_3 + \omega^3 \wedge \omega^2 \wedge v\omega^2{}_3 + \omega^1 \wedge \omega^2 \wedge \omega^3 (va^1{}_{32} - va^2{}_{31}).$$

In view of the above entered labels for the forms $\omega^1{}_3$, $\omega^3{}_1$, $\omega^3{}_2$ and $\omega^2{}_3$, and also representing $dv = v_i \omega^i$, we shall write:

$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^1 = \omega^1 \wedge \omega^2 \wedge (-vp_3 \omega^3) + \omega^1 \wedge \omega^3 \wedge v_2 \omega^2 + \omega^1 \wedge \omega^2 \wedge \omega^3 (va^2{}_{33}),$$
$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^2 = \omega^2 \wedge \omega^1 \wedge (vq_3 \omega^3) + \omega^2 \wedge \omega^3 \wedge v_1 \omega^1 + \omega^1 \wedge \omega^2 \wedge \omega^3 (-va^1{}_{33}),$$
$$- \omega^1 \wedge \omega^2 \wedge \omega^3 V^3 = \omega^3 \wedge \omega^1 \wedge (vq_2 \omega^2) + \omega^3 \wedge \omega^2 \wedge (-vp_1 \omega^1) + \omega^1 \wedge \omega^2 \wedge \omega^3 (va^1{}_{32} - va^2{}_{31}).$$

From here we have:

$$\begin{aligned} - V^1 &= - v - v_2 + va^2{}_{33}, \\ - V^2 &= - vq_3 + v_1 - va^1{}_{33}, \\ - V^3 &= vq_2 + vp_1 + va^1{}_{32} - va^2{}_{31}. \end{aligned} \tag{17}$$

From the last formula we have:

$$\frac{V^3}{v} = -\,(q_2 + p_1 + a^1{}_{32} - a^2{}_{31}). \qquad (18)$$

It is possible to show that the Gaussian curvature of the field of vectors $\mathbf{e_3}$ which is collinear to the vector of the blood velocity in each point of a vessel will be equal to

$$k_g = -\,p_2\, q_1 + q_1\, a^2{}_{32} - p_2\, a^1{}_{31} + a^2{}_{32}\, a^1{}_{31} - 1/4\,(p_1 - q_2 - a^1{}_{32} - a^2{}_{31})^2.$$

Let $d_1\mathbf{x}$ and $d_2\mathbf{x}$ be two transitions orthogonal to the vector of the field $\mathbf{e_3}$. By analogy with [10] a ratio of the volumes of parallelepipeds constructed on the triple $\mathbf{e_3}$, $\mathbf{e_3} + d_1\mathbf{e_3}$, $\mathbf{e_3} + d_2\mathbf{e_3}$ and on the triple $\mathbf{e_3}$, $d_1\mathbf{x}$, $d_2\mathbf{x}$ we shall name a total curvature of the field k_t in a point for the given subprojective space. Therefore

$$k_t = \frac{\mathbf{e}_3 \wedge (\mathbf{e}_3 + d_1\mathbf{e}_3) \wedge (\mathbf{e}_3 + d_2\mathbf{e}_3)}{\mathbf{e}_3 \wedge d_1\mathbf{x} \wedge d_2\mathbf{x}} = p_1\, q_2 + p_1\, a^1{}_{32} - p_2\, q_1 - p_2\, a^1{}_{31} - q_2\, a^2{}_{31} - a^2{}_{31}\, a^1{}_{32} +$$
$$+\, q_1\, a^2{}_{32} + a^1{}_{31}\, a^2{}_{32}.$$

Then $k_t - k_g = 1/4\,(p_1 + q_2 - (a^2{}_{31} - a^1{}_{32}))^2$ and thus

$$\sqrt{k_t - k_g} = 1/2\,|\,p_1 + q_2 + a^1{}_{32} - a^2{}_{31}|.$$

Comparing the latter with (18), we shall obtain:

$$\frac{|V^3|}{v} = |q_2 + p_1 + a^1{}_{32} - a^2{}_{31}| = 2\sqrt{k_t - k_g}\,. \qquad (19)$$

From (19) we can see that the ratio of the projection of a curl on the tangent of the streamline of blood to the magnitude of the blood velocity is some invariant of the blood streamline in the subprojective space.

Conclusions

The formulas for the determination of grad, div, rot in the subprojective space referred to the nonholonomic frame are obtained which enables to apply these formulas for the description of the turbulent blood flow. Such an approach allows to describe the blood flow in a special space: the subprojective space. The description of the flow of blood as well as any fluid in various spaces has more than once been considered by the authors of the given paper, and also by other authors [9]. Such an approach enables the application of rich geometric means for viewing the fluid flow [11] under various conditions, which allows, occasionally, not only to obtain the new properties of the flow, but also to model the human cardiovascular system in a new fashion.

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ГЕМОДИНАМИКА СЕРДЕЧНО-СОСУДИСТОЙ СИСТЕМЫ ЧЕЛОВЕКА ПРИ ДВИЖЕНИИ КРОВИ С ЗАВИХРЕНИЯМИ

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Данная работа посвящена моделированию движения крови с завихрениями, причем движение крови рассматривается в специальном римановом пространстве – субпроективном. Аналогом кровеносных сосудов в таком пространстве являются геодезические линии этого пространства. Описывается общий случай движения крови с завихрениями, при котором частица крови переходит из одной точки в другую по некоторому пути, который принадлежит неголономному распределению. С распределением связывается неголономный репер второго порядка. На такой основе описывается геометрия потока крови. Библ. 11.

Ключевые слова: гемодинамика, субпроективное пространство, геодезические линии, дифференциальные операторы

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