Russian Journal of Biomechanics, Vol. 4, № 3: 42-56, 2000

THE TORSION THEORY FOR THE HUMAN BONE

K.Ch. Wierzcholski

Department of Mechanical Engineering of Technical University of Szczecin, Piastow 19, PL-70781 Szczecin, Poland, e-mail: wunsch@safona.tuniv.szczecin.pl

Abstract: In this paper we consider the torsion theory for the human bone and the action of twisting moment on the joints which connect two bones. The thigh is loaded with the trunk forces, muscle forces, twisting moment and various couple forces. Bone cross section rarely has circular form. Therefore in this paper we solve the twisting problem for irregular, non-circular shapes of bone cross sections.

Key words: torsion theory of bone, loading of human joints

Introduction

Synovial joints of femoral bone utilize sliding between smooth spherical surfaces to enable a limb to be rotated while carrying a load. Fig. 1a shows the bone breaks, which are caused by the one-directional torsion moment [1]. Model of the thigh bone head and its hip joint loaded with torsion moment is shown additionally in Fig. 1b.

Many cases of femoral human bone breaks are caused by the couple forces, which are creating the torsion moment.

The present paper shows the theory of torsion for thigh human bones with circular cross sections and with non-circular, irregular transverse sections.

Basic equations

The complete system of equations of equilibrium of a homogeneous isotropic elastic bone without body forces is made up of the following equations [2]:

– equation of equilibrium:

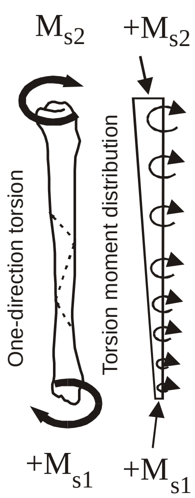


Fig. 1a. Loading of the human bone.

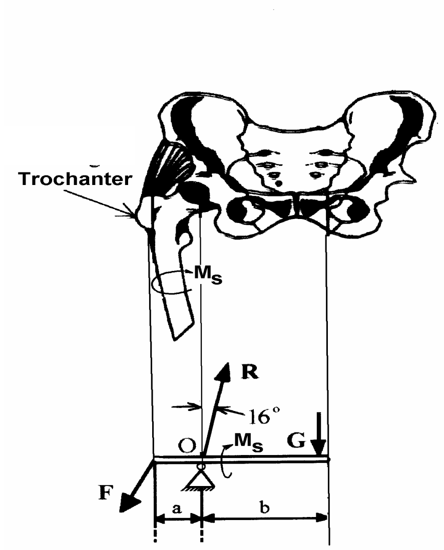


Fig. 1b. Loading model of the thigh bone head with torsion.

$$\tau_{ij,j} = 0, \text{ for } i, j \in (x, y, z)\,, \tag{1}$$

– stress-strain equations:

$$\tau_{ij} = \Lambda\delta_{ij}\varepsilon_{ii} + 2G\varepsilon_{ij}\,, \tag{2}$$

– strain-displacement relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{3}$$

Stress-strain relations for the viscoelastic properties of the bone have the following form [2]:

$$\tau_{ij} = \Lambda\delta_{ij}\varepsilon_{ii} + 2G\varepsilon_{ij} + 2\eta\frac{\partial}{\partial t}\left(\varepsilon_{ij}\right), \tag{4}$$

where τ_{ij} – components of the stress tensor, ε_{ij} -components of the strain tensor, δ_{ij} – components of the unit tensor, u_i – components of the displacement vector. Moreover Λ and G (shear modulus) denote Lame constants, η – dynamic viscosity of the bone material, t – time.

We substitute Eq. (3) into (4) and we obtain:

$$\tau_{ij} = \Lambda\delta_{ij}u_{i,i} + G\left(u_{i,j} + u_{j,i}\right) + \eta\frac{\partial}{\partial t}\left(u_{i,j} + u_{j,i}\right). \tag{5}$$

We expand equation (5), and hence we obtain:

$$\begin{aligned}
\tau_{xx} &= (\Lambda + G)\frac{\partial u_x}{\partial x} + 2\eta\frac{\partial^2 u_x}{\partial x\partial t}, \\
\tau_{yy} &= (\Lambda + G)\frac{\partial u_y}{\partial y} + 2\eta\frac{\partial^2 u_y}{\partial y\partial t}, \\
\tau_{zz} &= (\Lambda + G)\frac{\partial u_z}{\partial z} + 2\eta\frac{\partial^2 u_z}{\partial z\partial t}, \\
\tau_{zx} &= G\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) + \eta\left(\frac{\partial^2 u_z}{\partial x\partial t} + \frac{\partial^2 u_x}{\partial z\partial t}\right), \\
\tau_{zy} &= G\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) + \eta\left(\frac{\partial^2 u_z}{\partial y\partial t} + \frac{\partial^2 u_y}{\partial z\partial t}\right), \\
\tau_{xy} &= G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \eta\left(\frac{\partial^2 u_x}{\partial y\partial t} + \frac{\partial^2 u_y}{\partial x\partial t}\right).
\end{aligned} \tag{6}$$

The boundary conditions on the lateral surface of the bone cylinder have the following form:

$$\begin{aligned}
\tau_{xx}n_x + \tau_{xy}n_y &= 0, \\
\tau_{yx}n_x + \tau_{yy}n_y &= 0, \\
\tau_{zx}n_x + \tau_{zy}n_y &= 0,
\end{aligned} \tag{7}$$

where n_x and n_y are the components of the unit vector **n** which is normal to the lateral bone surface or normal to the curve C_0 which bounds transverse section of the bone.

Torsion of circular bone cross sections

Consider a circular bone cylinder of length *L*, with one of its bases fixed in the *xy*-plane, while the other base (in the plane *z* = *L*) is acted upon by a couple whose moment lies along the z-axis. Under the action of the couple, the bone will be twisted, and the generators of the bone cylinder will be deformed into helical curves. On account of the symmetry of the cross section, it is reasonable to suppose that sections of the bone cylinder by planes normal to the z-axis will remain plane after deformations and that the action of the couple will merely rotate each section through some angle Θ. Amount of rotation will clearly depend on the distance of the section from the base $z = 0$, and since the deformations are small, it is sensible to assume that the amount of rotation Θ is proportional to the distance of the section from the fixed base. Thus $\Theta = \alpha z$, where α is the twist per unit of bone length. If the cross sections of the bone cylinder remain plane after deformations, then the displacement u_z, along the z-axis, is zero. The displacements u_x, u_y, are readily calculated. Hence, consider any point $P(x,y)$ in the circular cross section, which, before deformations, occupied the position shown in Fig. 2.

After deformations the point P will occupy a new position $P(x+u_x, y+u_y)$. If the angle Θ is small we can write:

$$u_x = -\Theta\, y,\ u_y = \Theta\, x,\ \Theta = \alpha\, z\,, \tag{8}$$

hence we have for the displacements of any point with coordinates x,y,z:

$$u_x = -\alpha zy, \quad u_y = \alpha zx, \quad u_z = 0. \tag{9}$$

We put (9) into (6); thus we obtain:

$$\begin{gathered}\tau_{xx} = 0,\ \tau_{yy} = 0,\ \tau_{zz} = 0,\ \tau_{xy} = 0,\\ \tau_{zx} = -\alpha y(G + \eta \frac{1}{\alpha}\frac{\partial \alpha}{\partial t}),\\ \tau_{zy} = \alpha x(G + \eta \frac{1}{\alpha}\frac{\partial \alpha}{\partial t}).\end{gathered} \tag{10}$$

Stresses (10) obviously satisfy the equations of Equilibrium (1) with no body acting forces i.e.

$$\begin{gathered}\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0,\\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0,\\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0.\end{gathered} \tag{11}$$

The boundary conditions (7) on the lateral bone surface are likewise satisfied. The first two of Eqs.(7) are identically satisfied, and the last one gives:

$$\tau_{zx}\frac{x}{a} + \tau_{zy}\frac{y}{a} = 0 \tag{12}$$

for a circle of radius a and for $n_x = \cos(x,n) = x/a, \quad n_y = \cos(y,n) = y/a.$ Stresses (10) obviously satisfy Eq. (12).

Only M_z is the non-vanishing component of the couple **M** produced by the distribution of stresses (10) over the end of the bone cylinder, and it has the following form:

$$M_z = \iint_R \left(x\tau_{zy} - y\tau_{zx}\right)dxdy = G^*\alpha \iint_R \left(x^2 + y^2\right)dxdy = G^*\alpha\, J_0\,, \tag{13a}$$

where

$$G^* \equiv G + \eta\frac{\partial}{\partial t}\ln\alpha\,,\ \ J_0 \equiv \frac{\pi a^4}{2}. \tag{13b}$$

The magnitude of stress vector **T,** which lies in the plane of the bone section and is normal to the radius vector r joining point (x, y) with the origin (0,0), has the following form:

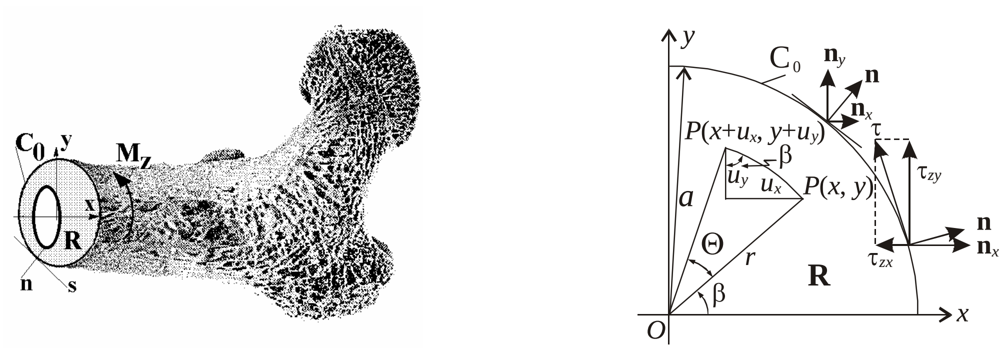


Fig.2 Circular transverse bone section.

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = G^* \alpha \sqrt{x^2 + y^2} = G^* \alpha\, r\,. \qquad (14)$$

From this we see that the maximum stress is a tangential stress that acts on the boundary of the bone cylinder and has the magnitude $G^* \alpha\, a$,where a is the radius of the bone cylinder.

Remark

For elastic bone properties α is independent of time t, hence $G = G^*$. Thus $G\alpha = \frac{M_z}{J_0}$. Stresses (10) are as follows: $\tau_{zx} = -\frac{yM_z}{J_0}$, $\tau_{zy} = \frac{xM_z}{J_0}$. This dependence is well-known in torsion theory of elasticity materials.

Torsion of long bones with non-circular cross section

The present subsection shows the theory of torsion for thigh human bones with non-circular, irregular transverse sections. Consider a bone subjected to no body forces and free from external forces on its lateral surface. One end of the bone is fixed by the joint in the plane $z = 0$, while the other end in the plane $z = L$ is twisted by a couple of magnitude M_z whose moment is directed along the axis of the bone [3] (see Fig.3). In Fig.3 symbol $\mathbf{n} \equiv \mathbf{i}\cos(x,\mathbf{n}) + \mathbf{j}\sin(y,\mathbf{n})$ denotes the exterior normal vector to the boundary C_0 of the cross section R of the bone, ds – boundary element of length, dx – element of length in x direction.

Theorem 1

If exists some harmonic function $\phi(x, y) \geq 0$ *in R whose integral of the normal derivative calculated over the entire arbitrary boundary* C_0 *of the cross bone section R, vanish, and if bone is free from external forces and is twisted by the couple of magnitude M whose moment is directed along the axis of the bone and is determined by the twist* α *per unit of bone length, then displacement and stresses have the following form:*

$$u_x = -\alpha z y,\; u_y = \alpha z x, \quad u_z = \alpha \cdot \phi(x, y)\,, \qquad (15)$$

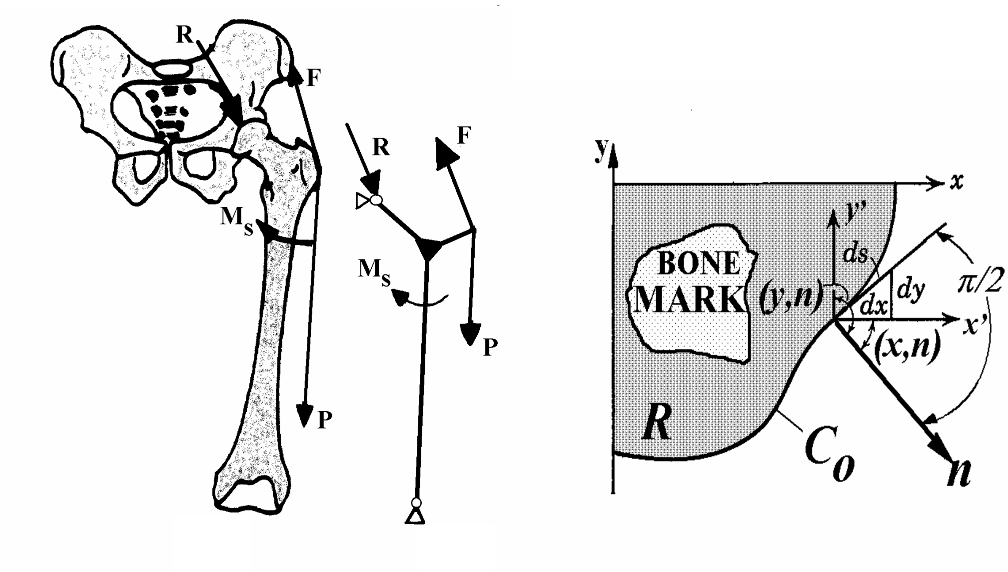
$$\tau_{yz} = G^* \alpha \left(\frac{\partial \phi}{\partial y} + x \right),\; \tau_{xz} = G^* \alpha \left(\frac{\partial \phi}{\partial x} - y \right),\; \tau_{xy} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0, \qquad (16)$$

In particular case for $\phi(\mathrm{x}, \mathrm{y}) \equiv 0$ *we can have only circular bone cross section R.*

Proof.

We put (15) into (6); thus we obtain stresses in the form (16).

Stresses (16) obviously satisfy the first two of Equilibrium Eqs.(11) with no body acting forces. The last one will be satisfied if function $\phi(x, y)$ satisfies the equation:



a b c

Fig.3. a) Maquet model for hip joint loading, b) statically scheme; **R –** trunk load force on the thigh bone head, **F** – force of the muscle; **P –** force of the hip-tibia band, $\mathbf{M_s}$ – twisting moment of the thigh bone, c) Non circular transverse section of the bone.

$$\Delta \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{17}$$

throughout the cross section R of the non-circular bone cylinder.

The first two of boundary conditions (7) on the lateral bone surface are identically satisfied, and the last one gives:

$$\left(\frac{\partial \phi}{\partial y} + x\right)\cos(y,n) + \left(\frac{\partial \phi}{\partial x} - y\right)\cos(x,n) = 0 \quad \text{on } C_0, \tag{18}$$

where C_0 is the boundary of the non-circular cross section R of the bone cylinder (see Fig. 3c). But on the ground of definition of normal vector and grad ϕ we have the following form of normal derivative:

$$\frac{d\phi}{dn} \equiv \frac{\partial \phi}{\partial x}\cos(x,\mathbf{n}) + \frac{\partial \phi}{\partial y}\cos(y,\mathbf{n}) = \mathbf{n}\ (\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y}) \equiv \mathbf{n}\ \text{grad}\,\phi \tag{19}$$

so that the boundary condition (18) by virtue of expression (19) can be written in the form:

$$\frac{d\phi}{dn} = y\cos(x,\mathbf{n}) - x\cos(y,\mathbf{n}) \quad \text{on } C_0. \tag{20}$$

The latter problem is associated with the name of Neumann and consists in determining a harmonic function $\phi(x,y)$ in a bone cross section region R whose normal derivative is prescribed on the boundary C_0 of the region. The condition for the existence of a solution $\phi(x,y)$ of the problem of Neumann is that the integral of the normal derivative of the function $\phi(x,y)$ calculated over the entire boundary C_0, vanish. This follows from the identity:

$$\int_{C_0} \frac{d\phi}{dn} ds = \int_{C_0} \mathbf{n}\ grad\phi\ ds = \iint_R div(grad\phi)dxdy = \iint_R \Delta\phi dxdy = 0, \tag{21}$$

where we substitute Eq. (19) into left-hand side, then we use Gauss Theorem in two-dimensional space and we put Eq. (17) in last term in the right-hand side. We denote *dx dy* as surface element. We can give the proof also when we put Eq. (20) in the left-hand side of Eq. (21):

$$\int_{C_0} \frac{d\phi}{dn} ds = \int_{C_0} [y\cos(x,\mathbf{n}) - x\cos(y,\mathbf{n})] ds . \tag{22}$$

But from Fig. 3c it is seen that tangential derivatives have the following form:

$$\frac{dx}{ds} = \cos(x,s) = \sin(x,n) = -\cos(y,n), \tag{23}$$

$$\frac{dy}{ds} = \sin(x,s) = \cos(x,n), \tag{24}$$

where n and s are the normal and tangential directions. We put Eqs. (23), (24) into Eq. (22) and we obtain:

$$\int_{C_0} \frac{d\phi}{dn} ds = \int_{C_0} (ydy + xdx) = 0. \tag{25}$$

From Green Theorem it follows that

$$\int_{C_0} \frac{d\phi}{dn} ds = \int_{C_0} (ydy + xdx) = \iint_R \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) dxdy = 0. \tag{26}$$

In particular case for $\phi(\mathrm{x},\mathrm{y}) \equiv 0$, condition (18) tends to equality:

$$x\cos(y,\mathbf{n}) - y\cos(x,\mathbf{n}) = 0 \text{ on } C_0 . \tag{27}$$

We substitute Eqs. (23), (24) into condition (27), hence we obtain equality:

$$xdx + ydy = 0. \tag{28}$$

This is the differential equation of a family of circles. Thus, in this case the considered bone must have only circular cross sections and C_0 denotes boundary circle. This fact completes the proof of Theorem 1♋.

<u>Corollary 1</u>

If the distribution of stresses is caused by the torsion of the bone only, without body forces, then resultant force acting on the end of the human cylindrical bone with the non-circular cross section vanishes.

<u>Proof</u>

The resultant forces in the *x*-direction are given by:

$$F_1 = \iint_R \tau_{zx} dxdy = G^*\alpha \iint_R \left(\frac{\partial\phi}{\partial x} - y \right) dxdy, \quad F_2 = \iint_R \tau_{zy} dxdy = G^*\alpha \iint_R \left(\frac{\partial\phi}{\partial y} + x \right) dxdy, \tag{29}$$

and by virtue of Eq. (17), first force F_1 can be written as:

$$F_1 = G^*\alpha \iint_R \left\{ \frac{\partial\phi}{\partial x} - y + x\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \right] \right\} dxdy. \tag{30}$$

We perform some algebraic transformations in integrand of Eq.(30), hence:

$$F_1 = G^*\alpha \iint_R \left\{ \frac{\partial}{\partial x}\left[x\left(\frac{\partial\phi}{\partial x} - y \right) \right] + \frac{\partial}{\partial y}\left[x\left(\frac{\partial\phi}{\partial y} + x \right) \right] \right\} dxdy. \tag{31}$$

Green's Theorem [4] is directly applicable to the integral (31), thus we get curvilinear integral:

$$F_1 = G^*\alpha \int_{C_0} - x\left(\frac{\partial\phi}{\partial y} + x \right) dx + x\left(\frac{\partial\phi}{\partial x} - y \right) dy . \tag{32}$$

Now we substitute dependencies (23), (24) into curvilinear integral (32), hence we have:

$$F_1 = G^*\alpha \int_{C_0} \left\{ x\left(\frac{\partial\phi}{\partial y} + x\right)\cos(y,n) + x\left(\frac{\partial\phi}{\partial x} - y\right)\cos(x,n) \right\} ds. \tag{33}$$

After transformations of the integrand of integral (33) we obtain:

$$F_1 = G^*\alpha \int_{C_0} \left\{ \frac{\partial\phi}{\partial y}\cos(y,n) + \frac{\partial\phi}{\partial x}\cos(x,n) + x\cos(y,n) - y\cos(x,n) \right\} xds. \tag{34}$$

Making use of expression (19), we rewrite integral (34) in the form:

$$F_1 = G^*\alpha \int_{C_0} \left\{ \frac{d\phi}{dn} + x\cos(y,n) - y\cos(x,n) \right\} xds. \tag{35}$$

Now we substitute dependencies (20) into curvilinear integral (35), thus we have:

$$F_1 = G^*\alpha \int_{C_0} \left\{ \frac{d\phi}{dn} - \frac{d\phi}{dn} \right\} xds = 0. \tag{36}$$

It may be shown in similar way that second force F_2 in formula (29) vanishes too, so that the resultant force acting on the end of the bone vanishes. This fact completes the proof of Corollary ♋.

Corollary 2

The system of stresses defined by Eqs.(16) and caused by the torsion of the bone only, without body forces, is statically equivalent to a torsional couple, where resultant moment of the external forces applied to the end of the bone has the following form [5]:

$$M_z = \iint_R \left(x\tau_{zy} - y\tau_{zx} \right) dxdy = G^*\alpha \iint_R \left(x^2 + y^2 + x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \right) dxdy \tag{37}$$

*and the magnitude of stress vector **T** which lies in the plane of the bone section is as follows:*

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = G^*\alpha \sqrt{\left(\frac{\partial\phi}{\partial x} - y\right)^2 + \left(\frac{\partial\phi}{\partial y} + x\right)^2}\ . \tag{38}$$

Proof

If we substitute stresses (16) in the left-hand side of the formula (37), then after the transformations we obtain the right-hand side of Eq. (37). The integral appearing in (37) depends on the torsion function $\phi(x,y)$ and bone cross section *R* and provides a measure of the rigidity of a bone subjected to torsion.

If we substitute stresses (16) in the left hand of expression (38), then the magnitude of stress vector **T** which lies in the plane of the bone section and is normal to the radius vector **r** joining point (*x*,*y*) with the origin (0,0), has the form of right-hand side of equation (38).

In particular case for $\phi(x,y) = 0$ i.e. for bone with circular cross section, the formula (38) has the form (14) what completes the proof of Corollary 2♋.

Corollary 3

If exists some harmonic function $\psi(x,y) \geq 0$:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \quad in\ \ R \tag{39}$$

which over the entire arbitrary boundary C_0 *of the cross bone section R, has the following form:*

$$\psi(x,y)=\frac{1}{2}\left(x^2+y^2\right)+const \text{ on } C_0 \tag{40}$$

and if bone is free from external forces and is twisted by the couple of magnitude M whose moment is directed along the axis of the bone and is determined by the twist α *per unit of bone length, then stresses are as follows:*

$$\tau_{yz}=G^*\alpha\left(-\frac{\partial\psi}{\partial x}+x\right),\ \tau_{xz}=G^*\alpha\left(\frac{\partial\psi}{\partial y}-y\right),\ \tau_{xy}=\tau_{xx}=\tau_{yy}=\tau_{zz}=0. \tag{41}$$

Moreover resultant moment of the external forces applied to the end of the bone presents the following expression:

$$M_z=\iint_R\left(x\tau_{zy}-y\tau_{zx}\right)dxdy=G^*\alpha\iint_R\left(x^2+y^2-x\frac{\partial\psi}{\partial x}-y\frac{\partial\psi}{\partial y}\right)dxdy. \tag{42}$$

and the magnitude of stress vector ***T*** *that lies in the plane of the bone section, shows the dependence:*

$$\tau=\sqrt{\tau_{zx}^2+\tau_{zy}^2}=G^*\alpha\sqrt{\left(\frac{\partial\psi}{\partial y}-y\right)^2+\left(\frac{\partial\psi}{\partial x}-x\right)^2}\ . \tag{43}$$

Proof

Since the torsion function $\phi(x,y)$ is harmonic in the region R representing the cross-section of the bone, one can construct the analytic function $\phi(x,y)+\mathrm{i}\ \psi(x,y)$ of complex variable $x+iy$ where $\psi(x,y)$ is the conjugate harmonic function, related to $\phi(x,y)$ by the Cauchy - Riemann equations [3]:

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y},\quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \tag{44}$$

Since the function $\phi(\mathrm{x},\mathrm{y})+\mathrm{i}\ \psi(\mathrm{x},\mathrm{y})$ is an analytic function, we have [3]:

$$\psi(x,y)=\int_{C_0}\left(\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy\right)=\int_{P_0(x_0,y_0)}^{P(x,y)}\left(-\frac{\partial\phi}{\partial y}dx+\frac{\partial\phi}{\partial x}dy\right), \tag{45}$$

where the integral is evaluated over an arbitrary path joining some point $P_0(x,y)$ with a arbitrary point P(x,y) belonging to the region R. If the region R is simply connected, the function $\psi(x,y)$ will be single valued. If the region R of bone cross section is multiple-connected, then $\psi(x,y)$ may turn out to be multiple-valued. Now we can show that our torsion bone problem leads to the determining a function $\psi(x,y)$ that satisfies the equation (39) and that satisfies the boundary condition (40).

We can write the expression for the normal derivative (19) with the aid of the tangential derivatives (23), (24) in following form:

$$\frac{d\phi}{dn}=\frac{\partial\phi}{\partial x}\cos(x,n)+\frac{\partial\phi}{\partial y}\cos(y,n)=\frac{\partial\phi}{\partial x}\frac{dy}{ds}-\frac{\partial\phi}{\partial y}\frac{dx}{ds}. \tag{46}$$

Now we substitute Cauchy – Riemann dependencies (44) into formula (46), hence we have:

$$\frac{d\phi}{dn}=\frac{\partial\phi}{\partial x}\frac{dy}{ds}-\frac{\partial\phi}{\partial y}\frac{dx}{ds}=\frac{\partial\psi}{\partial x}\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}\equiv\frac{d\psi}{ds}. \tag{47}$$

Additionally we transform expression for the normal derivative (20) with the aid of the tangential derivatives (23), (24) in following form:

$$\frac{d\phi}{dn}=y\cos(x,\mathbf{n})-x\cos(y,\mathbf{n})=x\frac{dx}{ds}+y\frac{dy}{ds}=\frac{1}{2}\frac{d}{ds}\left(x^2+y^2\right). \tag{48}$$

We equate right-hand sides of equations (47) and (48), hence

$$\frac{d\psi}{ds} = \frac{1}{2}\frac{d}{ds}\left(x^2 + y^2\right). \quad (49)$$

Thus after the integration we obtain function $\psi(x, y)$ in the form (40). This fact completes the first step of proof of corollary 3.

Making use of the Cauchy – Riemann conditions (44) in expressions (14), we obtain stresses in the form (40), what completes the second step of the proof of corollary 3.

Making use of the Cauchy – Riemann conditions (44) in expressions (37), (38), we obtain resultant moment and stress vector in the form (42), (43) what completes the total proof of corollary 3 ♋.

Corollary 4

If exists some function:

$$\Psi(x, y) \equiv \psi(x, y) - \frac{1}{2}\left(x^2 + y^2\right), \quad (50)$$

which satisfies following Poisson Equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = -2, \ \ in \ R \quad (51)$$

and which over the entire arbitrary boundary C_0 of the cross bone section R, has the following form:

$$\Psi = const \ \text{on} \ C_0, \quad (52)$$

and if the bone is free from external forces and is twisted by the couple of magnitude M whose moment is directed along the axis of the bone and is determined by the twist α per unit of bone length, then stresses show the following expressions:

$$\tau_{yz} = -G^*\alpha\frac{\partial\Psi}{\partial x}, \ \tau_{xz} = G^*\alpha\frac{\partial\Psi}{\partial y}, \ \ \tau_{xy} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0. \quad (53)$$

Moreover resultant moment of the external forces applied to the end of the bone presents the dependence:

$$M_z = \iint\limits_R \left(x\tau_{zy} - y\tau_{zx}\right)dxdy = -G^*\alpha\iint\limits_R \left(x\frac{\partial\Psi}{\partial x} + y\frac{\partial\Psi}{\partial y}\right)dxdy. \quad (54)$$

*and the magnitude of stress vector **T** which lies in the plane of the bone section is as follows:*

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = G^*\alpha\sqrt{\left(\frac{\partial\Psi}{\partial y}\right)^2 + \left(\frac{\partial\Psi}{\partial x}\right)^2} \ . \quad (55)$$

Proof

Two first and two second partial derivatives of expression (50) give:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\Psi}{\partial x} + x, \ \ \frac{\partial\psi}{\partial y} = \frac{\partial\Psi}{\partial y} + y, \quad (56)$$

$$\frac{\partial^2\psi}{\partial \mathrm{x}^2} = \frac{\partial^2\Psi}{\partial \mathrm{x}^2} + 1, \ \ \frac{\partial^2\psi}{\partial \mathrm{y}^2} = \frac{\partial^2\Psi}{\partial \mathrm{y}^2} + 1. \quad (57)$$

If we substitute expressions (57), (50) into equations (39), (40), then we obtain dependencies (51), (52). If we substitute expression (56) into equation (41), then we obtain dependence (53). If we substitute expression (56) into equations (42), (43), then we obtain dependencies (54), (55) for resultant moment on the end of the bone and stress vector **T** which lies in the plane of the bone section. These remarks complete the proof of corollary 4 ♋.

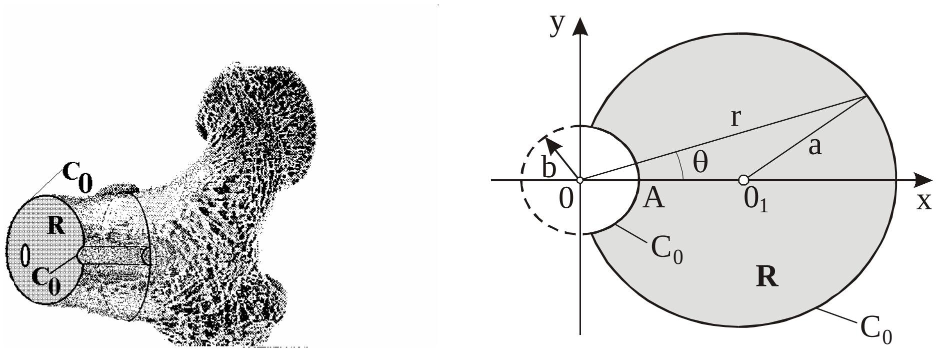


Fig. 4. Bone cross section with the groove.

The torsion of the bone with groove

The effect of grooves in bone (see Fig.4) on the maximum shearing stress can be discussed in an elementary way by studying an example due to C. Weber [5].

In the polar coordinates defined by the equations $x = r\cos\theta,\ y = r\sin\theta$ we can construct the following harmonic function:

$$\psi = a\left(x - b^2\frac{x}{x^2+y^2}\right) + \frac{1}{2}b^2 = a\cos\theta\left(r - \frac{b^2}{r}\right) + \frac{1}{2}b^2 \quad \text{on } R \tag{58}$$

where a and b are constant.

On the boundary C_0 of the bone cross section, function (58) must reduce to:

$$\psi = \frac{1}{2}\left(x^2+y^2\right) = \frac{1}{2}r^2 \quad \text{on } C_0. \tag{59}$$

We equate the right-hand sides of equations (58), (59), hence we obtain equation of the boundary for which the function ψ solves the torsion problem:

$$a\left(r\cos\theta - \frac{b^2\cos\theta}{r}\right) + \frac{1}{2}b^2 = \frac{1}{2}r^2. \tag{60}$$

After the factorization expression (60) takes the following form:

$$\left(r^2 - b^2\right)\left(1 - \frac{2a\cos\theta}{r}\right) = 0, \tag{61}$$

for $b \le r \le 2a\cos\theta$. Thus the boundary is made up of two circles: $r = b$ and $r = 2a\cos\theta$ which are shown in Fig. 4.

If we put harmonic function (58) in dependencies of stresses (41), and if we use derivatives of a composite function:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x}, \quad \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y}, \tag{62}$$

and derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, \quad \frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}, \tag{63}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \quad \frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r},$$

then we obtain following stresses in bone cross section in polar coordinates:

$$\tau_{zx} = G^*\alpha a\left(\frac{k^2}{r_1^2}\sin 2\theta - r_1\sin\theta\right) \quad \text{and} \quad \tau_{zy} = G^*\alpha a\left(r_1\cos\theta - 1 - \frac{k^2}{r_1^2}\right) \tag{64}$$

for $k \le r_1 \equiv \dfrac{r}{a} \le 2\cos\theta$ and $k \equiv \dfrac{b}{a}$. Additionally for $r = b$ i.e. on the surface of the groove the shear stresses in polar coordinates have the following form:

$$\tau_{zx}^{(b)} = G^* \alpha a(2\cos\theta - k)\sin\theta, \quad \tau_{zy}^{(b)} = -G^* \alpha a(2\cos\theta - k)\cos\theta, \qquad k \equiv \frac{b}{a}. \tag{65}$$

It is easy to see that

$$\lim_{r_1 \to k} \tau_{zy} \Rightarrow \tau_{zy}^{(b)} \quad \text{and} \quad \lim_{r_1 \to k} \tau_{zx} \Rightarrow \tau_{zx}^{(b)} \quad . \tag{66}$$

Magnitude of tangential stresses on the surface of the groove is given by:

$$\tau^{(b)} = \sqrt{\left(\tau_{zx}^{(b)}\right)^2 + \left(\tau_{zy}^{(b)}\right)^2} = G^* \alpha a(2\cos\theta - k), \tag{67}$$

where $|\theta| \le \theta_0 \equiv \arccos\left(\dfrac{k}{2}\right)$ for $k > 0$ and $\theta \equiv \theta_o = \dfrac{\pi}{2}$ for $k = 0$.

The maximum shearing stress is at the deepest point $A(b,0)$ of the groove i.e. for the angle $\theta = 0$. At this point the shear stress (67) has the value:

$$\tau_{\max}^{(b)} = 2aG^* \alpha\left(1 - \frac{k}{2}\right) \quad \text{for } 0<b<2a, \ 0<k<2. \tag{68}$$

If we put harmonic function (58) in dependencies of stress (41), and if we put there the derivatives of a composite function (62), (63) then for $r = 2a\cos\theta$ i.e. on the lateral surface of the bone without the groove, we obtain following stresses in polar coordinates:

$$\tau_{zx}^{(a)} = G^* \alpha \frac{a}{4}\left(k^2 - 4\cos^2\theta\right)\frac{\sin 2\theta}{\cos^2\theta}, \quad \tau_{zy}^{(a)} = -G^* \alpha \frac{a}{4}\left(k^2 - 4\cos^2\theta\right)\frac{\cos 2\theta}{\cos^2\theta}. \tag{69}$$

It is easy to see that

$$\lim_{r_1 \to 2\cos\theta} \tau_{zy} \Rightarrow \tau_{zy}^{(a)} \quad \text{and} \quad \lim_{r_1 \to 2\cos\theta} \tau_{zx} \Rightarrow \tau_{zx}^{(a)}. \tag{70}$$

Magnitude of tangential stresses on the lateral surface of the bone without the groove is given by:

$$\tau^{(a)} = \sqrt{\left(\tau_{zx}^{(a)}\right)^2 + \left(\tau_{zy}^{(a)}\right)^2} = G^* \alpha a\left(1 - \frac{k^2}{4}\sec^2\theta\right), \tag{71}$$

where $|\theta| \le \theta_0 \equiv \arccos\left(\dfrac{k}{2}\right)$ for $k > 0$ and $\theta \equiv \theta_o = \dfrac{1}{2}\left(\dfrac{\pi}{2} - \dfrac{\pi}{2}\right) = 0$ for $k = 0$.

The maximum shearing stress is at the point ($2a$, 0) i.e. for the angle $\theta = 0$. At this point the shear stress (71) has the value:

$$\tau_{\max}^{(a)} = aG^* \alpha\left(1 - \frac{k^2}{4}\right) \quad \text{for } 0<b<2a, \ 0<k<2. \tag{72}$$

The ratio of the maximum shearing stresses at the groove and lateral bone surface has the values in the following interval:

$$1 < \frac{\tau_{\max}^{(b)}}{\tau_{\max}^{(a)}} = \frac{2}{1 + \dfrac{k}{2}} \le 2, \quad \text{for} \quad 0 < b < 2a, \quad 0 < k < 2. \tag{73}$$

It is easy to see that at the edge of bone excision at the point

$$\theta = \pm\theta_0, \ r = b \quad \text{i.e. for} \quad x = \frac{b^2}{2a}, \ y = \pm b\sqrt{1 - \left(\frac{b}{2a}\right)^2} \tag{74}$$

the shear stresses are equal to zero:

$$\tau^{(a)}(\theta = \pm\theta_0 \ne 0) = 0, \quad \tau^{(b)}(\theta = \pm\theta_0 \ne 0) = 0 \quad \text{for} \quad k > 0 \tag{75}$$

and

$$\tau^{(a)}(\theta \equiv \theta_0 = 0) = G^* \alpha a, \quad \tau^{(b)}(\theta \equiv \theta_0 = 0) = 0 \quad \text{for} \quad k = 0.$$

We substitute the derivatives in x and y directions of a composite function (62), (63) into the resultant moment (42) of the external forces occurring at the end of the bone. After calculations in polar co-ordinates we obtain the following form:

$$M_z = G^* \alpha \iint_{R(x,y)} \left(x^2 + y^2 - x\frac{\partial \psi}{\partial x} - y\frac{\partial \psi}{\partial y} \right) dxdy =$$

$$= G\alpha \iint_{R(r,\theta)} \left[r^2 - \left(\frac{\partial \psi}{\partial r}\frac{dr}{dx} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial x} \right) r\cos\theta - \left(\frac{\partial \psi}{\partial r}\frac{dr}{dy} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial y} \right) r\sin\theta \right] rd\theta dr =$$

$$= G\alpha \iint_{R(r,\theta)} \left[r^2 - \frac{\partial \psi}{\partial r} r\cos^2\theta + \frac{\partial \psi}{\partial \theta}\sin\theta\cos\theta - \frac{\partial \psi}{\partial r} r\sin^2\theta - \frac{\partial \psi}{\partial \theta}\sin\theta\cos\theta \right] \quad rd\theta dr =$$

$$= G\alpha \iint_{R(r,\theta)} \left(r^2 - r\frac{\partial \psi}{\partial r} \right) rd\theta dr, \tag{76}$$

for domain $R(r,\theta): \{-\theta_0 \le \theta \le +\theta_0, \quad b \le r \le 2a\cos\theta\}$ where it follows from the equation (58) that

$$\frac{\partial \psi}{\partial r} = \left(1 + \frac{b^2}{r^2} \right) a\cos\theta. \tag{77}$$

After integration by means of the iterated integral in domain R, we obtain the following dependencies:

$$M_z = G^* \alpha \iint_{R(r,\theta)} \left[r^2 - \left((1 + \frac{b^2}{r^2} \right) ra\cos\theta \right] rd\theta dr =$$

$$= G^* \alpha \int_{-\theta_0}^{+\theta_0} \left[\int_b^{2a\cos\theta} \left(r^3 - ar^2\cos\theta - ab^2\cos\theta \right) dr \right] d\theta =$$

$$= G^* \alpha \int_{-\theta_0}^{+\theta_0} \left(\frac{4}{3}a^4\cos^4\theta - 2a^2b^2\cos^2\theta + \frac{4}{3}ab^3\cos\theta - \frac{1}{4}b^4 \right) d\theta. \tag{78}$$

Finally, the value of resultant moment takes the form:

$$\mathrm{M_z} = \mathrm{G}^* \alpha \mathrm{a}^4 \left[\left(1 - 2\mathrm{k}^2 - \frac{1}{2}\mathrm{k}^4 \right) \arccos\left(\frac{\mathrm{k}}{2} \right) + \left(\frac{7}{4}\mathrm{k}^3 + \frac{1}{2}\mathrm{k} \right) \sqrt{1 - \left(\frac{\mathrm{k}}{2} \right)^2} \right], \tag{79}$$

for $0 \le b < 2a, \quad 0 \le k < 2.$

It is easy to see that for $b = 0$, i.e. $k = 0$, the moment (79) tends to the simple value (13):

$$\lim_{k \xrightarrow{b=0} 0} M_z = \frac{1}{2}\pi G^* \alpha a^4. \tag{80}$$

Numerical calculations

Now by virtue of expressions (67) and (71) we examine numerically dimensionless values of resultant shear stresses on the boundary of bone cross section with groove for various ratio $k = b/a$, but in clinical practice k ranges from 0.05 to 0.10. To obtain the dimensional value of shear stresses, we must multiply the dimensionless values by the dimensional factor $G\alpha a$. We can show, that even small groove (k = 0.10) gives the very large tangential stresses on the surface of the groove (see Fig. 5).

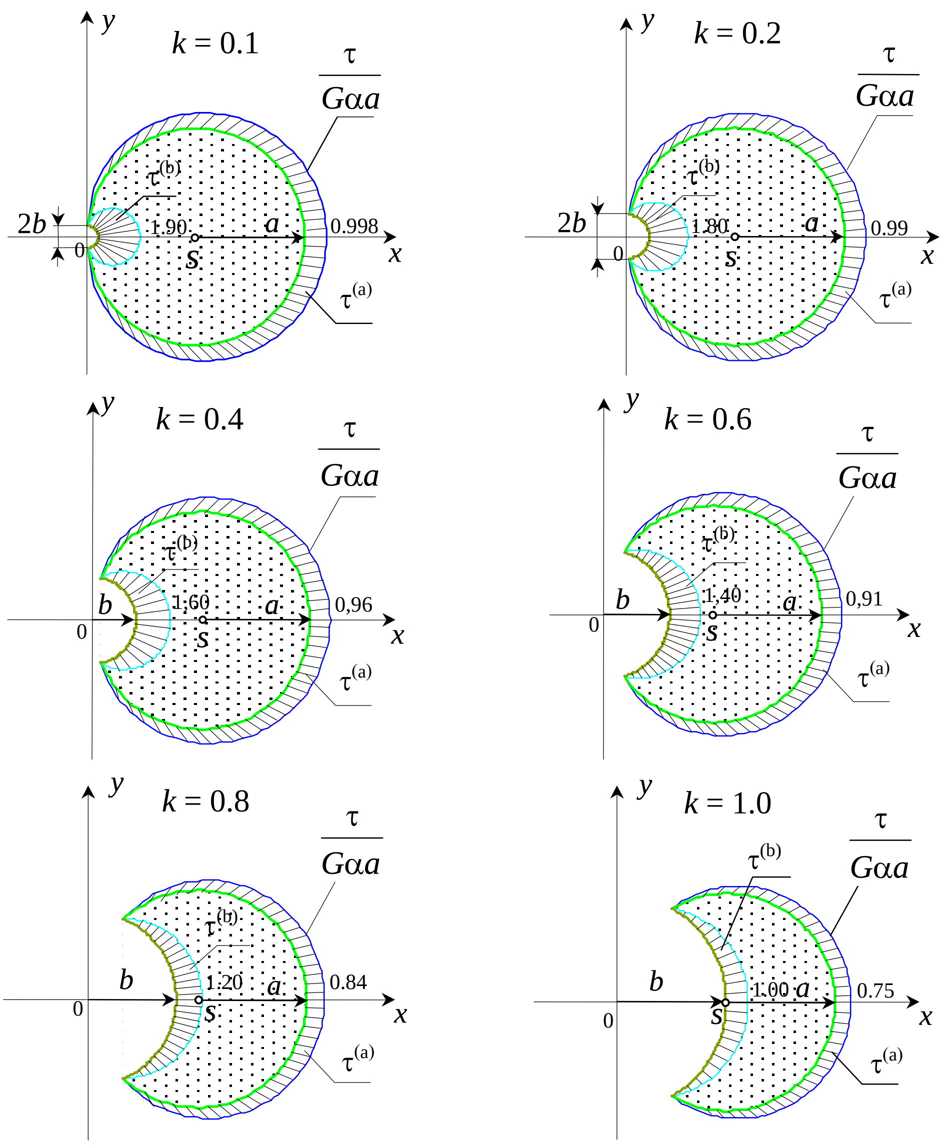


Fig. 5. Resultant shear stresses on the boundary of the bone cross section with groove.

Fig. 6 shows the dimensionless contour lines of both tangential stresses τ_{zy} and τ_{zx} occurring in the neighbourhood of the groove and in the cross section of the bone. To obtain real values of shear stresses, we must multiply the dimensionless values of shear stresses indicated in Fig.6 by the dimensional factor: $G\alpha a$ in N/m^2.

Conclusions

The present paper shows the preliminary considerations of the twisting theory of human bones with circular and non-circular cross sections by means of the Neumann, Dirichlet and Prandtl boundary problems. Some Corollaries concerning the bone twisting and joints are defined, where values of shear stresses, resultant moment and the resultant magnitude of stress vector are indicated. Moreover the presented theory is applied to solving of the problem of the torsion of the bone with groove. In particular cases, all obtained results tend to the simple solution, which is well-known in classical torsion theory.

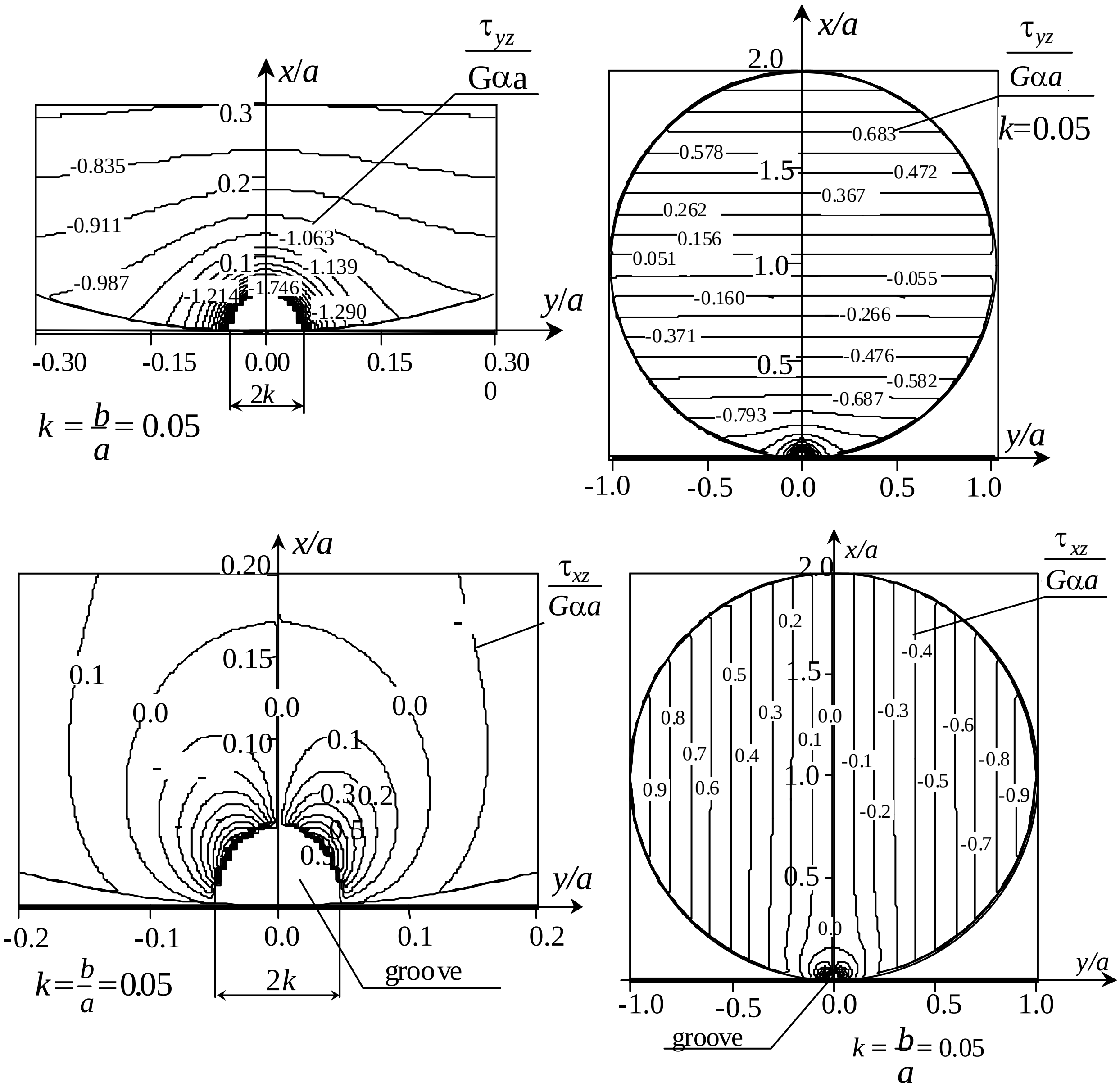


Fig. 6. Tangential stresses τ_{zx} and τ_{zy} in the neighborhood of the groove and in the total region of the bone cross section.

Acknowledgement

Author thanks for the financial support by the Polish Grant KBN 8-T 11E-021-17.

References

1. WIERZCHOLSKI K. **The thigh human bone torsion with hollow canals**, Tribologia, 997-1007, 1998.
2. NOWACKI W. **Theory of Elasticity**, Warsaw, PWN, 1970.
3. Maquet P.G.J. **Biomechanics of the hip**. Springer Verlag Heidelberg, Berlin, 1985.
4. LEJA F. **Funkcje zespolone**. Warszawa, PWN Biblioteka Matematyczna, 1970 (in Polish).
5. Sokolnikoff I.S. **Mathematical Theory of Elasticity**. McGraw-Hill Book Company, INC, New York, 1956.

ТЕОРИЯ КРУЧЕНИЯ КОСТЕЙ ЧЕЛОВЕКА

К.Х. Вежхольский (Щецин, Польша)

Рассмотрена теория кручения костей человека при воздействии крутящего момента на суставы, соединяющего кости. В частности, бедренная кость испытывает действие сил со стороны туловища, мышечных сил, крутящего момента и распределенной системы моментов. Так как поперечное сечение кости редко имеет круговую форму, то представляет интерес решение проблемы кручения для поперечных сечений кости нерегулярной формы. Решение проводится для моделей упругого и вязкого поведения материала кости.

Решение задачи приводит к краевым задачам Неймана, Дирихле и Прандтля. Получен ряд следствий о свойствах моментов, сил и напряжений при кручении. Кроме того, рассмотрено обобщение теории для кручения кости с выемкой. В частных случаях полученные результаты приводят к известным результатам классической теории кручения. Библ 5.

Ключевые слова: кручение, трубчатая кость, суставы, крутящий момент, аналитическое решение

Received 05 May 2000