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ON THE THEORY OF BENDING OF FOOT PROSTHESIS CONTAINING THE CURVED PLATES

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Abstract: The joint cylindrical bending of two plates which are curved in their natural states and are fixed at one end is considered. It is assumed that the load is localized in the projecting part of the lower (longer) plate. It is proved that the contact of the plates under the load takes place only at one point (except for the fixed point) at the end of the upper (shorter) plate. This result may be used in the investigation of the framework of the foot prosthesis representing a leaf spring.

Key words: foot prosthesis, leaf spring, curved plates, weak bending, contact area

Introduction

The problem of the joint cylindrical bending of two plates which are flat in their natural states, has been solved in [1]. This problem arises in the investigation of the framework of the foot prosthesis representing a leaf spring. It follows, in particular, from the results of [1] that if the load applied to the lower plate is localized only in the projecting part of this plate (Fig. 1) then the contact of the plates under bending takes place only at one point (except for the fixed point; Fig. 1; this statement is the particular case of the item (ii) of the theorem 1 in [1]).

It is known [2, 3] that in some foot prosthesis designs the plates of the framework are curved in their natural states. Therefore in the present study the above-mentioned statement is proved (under some conditions) for the curved plates.

Formulation of the problem

The natural shape of the plates will be described by the function $\varphi(x)$ [2, 3], where x is the length of the (curvilinear) segment of the plate placed between the fixed point and some arbitrary point; φ is the angle formed by the tangents to the plate at these points (Fig. 2); $0 \le x \le \ell$; ℓ is the length of the plate.

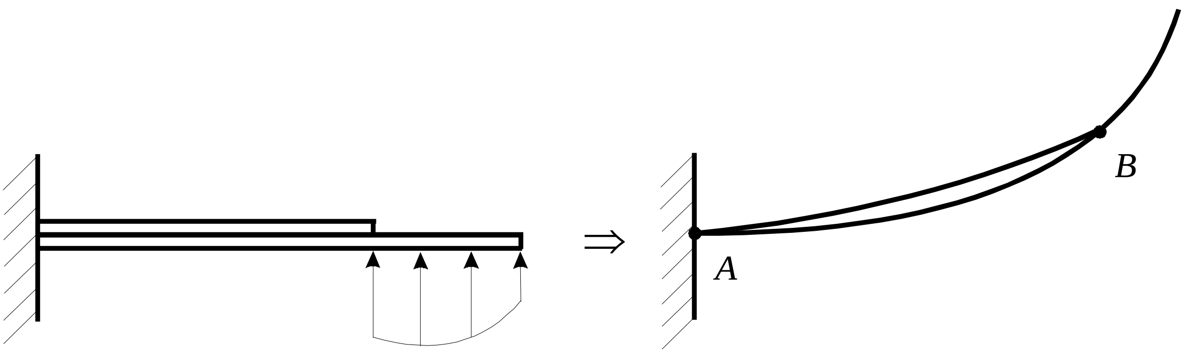

Fig. 1. The contact of the plates takes place at the points *A* and *B*.

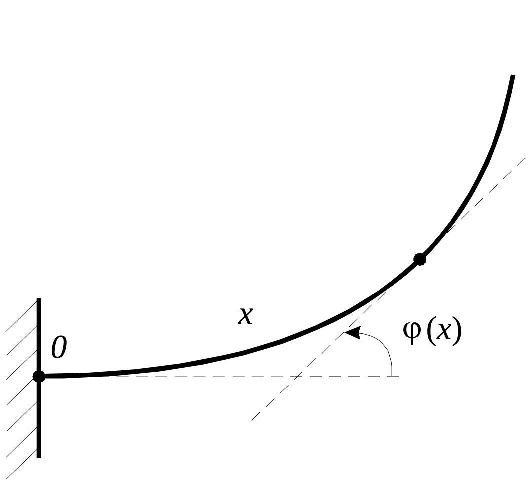

Fig. 2. Definition of the function $\varphi(x)$.

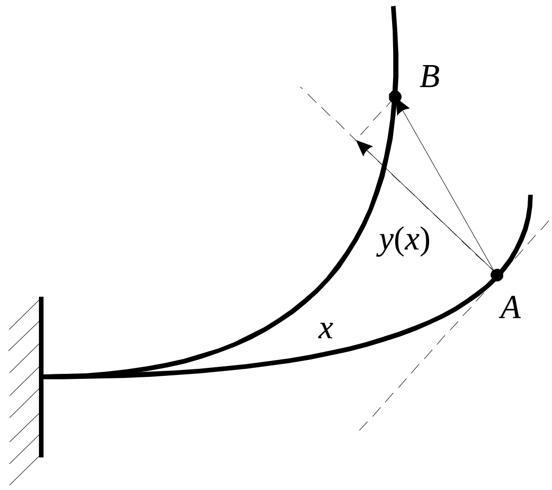

Fig. 3. Definition of the function $y(x)$.

The shape of the plate under the load will be described by the analogous function $\tilde{\varphi}(x)$. It is assumed that the potential energy stored in the plate under bending may be expressed as

$$P = \frac{1}{2c}\int_0^{\ell} \left(\tilde{\varphi}'(x) - \varphi'(x)\right)^2 dx, \tag{1}$$

where $c = 12(1-\sigma^2)/Edh^3$, E is the Young's modulus, σ is the Poisson's ratio, d is the width of the plate, h is its thickness [4]. It is assumed that the bending of the plates is weak (linear approximation with respect to the load). In this case the shape of the plate under the load may be defined also by the normal displacements $y(x)$ (Fig. 3). Let A be the point of the plate without bending with the curvilinear coordinate x; let B be the position of the same point of the plate under bending. Then $y(x)$ is the projection of the vector $\overline{AB}$ onto the normal to the plate at the point A.

It is assumed that the load is perpendicular to the plate (Fig. 4); $q(x)$ is the load density. Using (1), the principle of virtual displacements and the standard calculus of variations techniques [5], one can find that

$$y(x) = c\int_0^{\ell} G(x,\xi) q(\xi) d\xi, \tag{2}$$

where

$$G(x,\xi) = \int_0^{\min(x,\xi)} g(\eta, x) g(\eta, \xi) d\eta, \tag{3}$$

$$g(\eta,\zeta) = \int_{\eta}^{\zeta} \cos(\varphi(\zeta) - \varphi(\mu)) d\mu. \tag{4}$$

Consider the joint bending of two plates with the same natural shapes. It is assumed that the load is localized only in the projecting part of the lower plate (Fig. 5). The lengths of the lower (№ 1) and the upper (№ 2) plates are denoted as L and ℓ, respectively. The plates are impenetrable to each other; this constraint may be formulated as

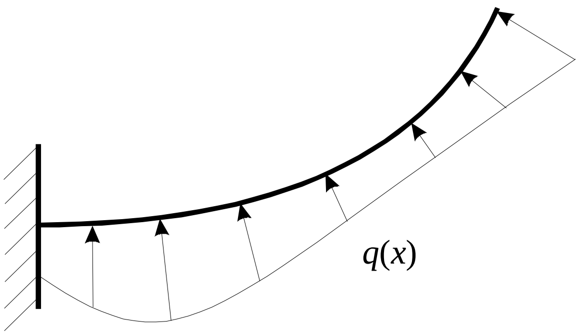

Fig. 4. The load is perpendicular to the plate.

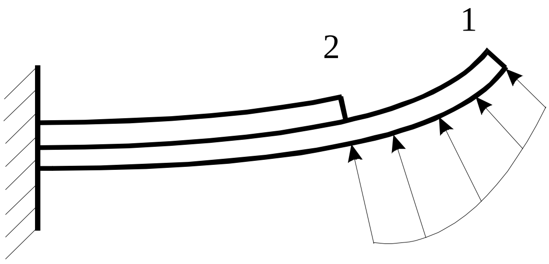

Fig. 5. The scheme of the joint bending of two curved plates.

$$y_2(x) \geq y_1(x) \text{ for } 0 \leq x \leq \ell . \tag{5}$$

Problem.

It is required to prove that the scheme of bending shown in Fig. 5, corresponds to the bending pattern shown in Fig. 1, i.e. the contact of the plates under bending takes place only at one point (except for the fixed point).

Solution of the problem

The formula (2) is analogous to the formula (2) in [1] (but the function G is different), and the constraint (5) and the corresponding constraint in [1, p. 13] coincide. Thus, repeating the calculations [1, pp. 12-14], one can find that the solution of the problem is reduced to the proof of inequality $R(\ell, x, \xi) \geq 0$ where

$$R(\ell, x, \xi) = G(x, \ell)G(\ell, \xi) - G(\ell, \ell)G(x, \xi) ; \tag{6}$$

$0 \leq x \leq \ell \leq \xi \leq L$. (One should consider $\xi \geq \ell$ in the formula (13) in [1], because it is assumed in the present study that $q(x) = 0$ for $0 \leq x < \ell$.)

Lemma.

If the function $\varphi(x)$ is non-decreasing and $0 \leq \varphi(x) \leq \pi/2$ for $0 \leq x \leq L$ then $g(\beta,\delta)g(\alpha,\gamma) - g(\beta,\gamma)g(\alpha,\delta) \geq 0$ for $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq L$.

Proof.

Using (4), we find:

$$g(\beta,\delta)g(\alpha,\gamma) - g(\beta,\gamma)g(\alpha,\delta) = I_1 + I_2 - I_3 \tag{7}$$

where

$$I_k = \iint_{D_k} A(\gamma,\delta,\mu,\nu)\, d\mu\, d\nu \; ; \; k = 1, 2, 3 ;$$

$$D_1 : (\gamma \leq \mu \leq \delta, \alpha \leq \nu \leq \beta), \; D_2 : (\beta \leq \mu \leq \gamma, \alpha \leq \nu \leq \beta), \; D_3 : (\alpha \leq \mu \leq \beta, \beta \leq \nu \leq \gamma),$$

$$A(\gamma,\delta,\mu,\nu) = \cos(\varphi(\delta) - \varphi(\mu))\cos(\varphi(\gamma) - \varphi(\nu)) .$$

One can easily find that

$$I_1 \geq 0 . \tag{8}$$

Changing over the variables of integrating in I_3, we obtain

$$I_2 - I_3 = \sin(\varphi(\delta) - \varphi(\gamma)) \iint_{D_2} \sin(\varphi(\mu) - \varphi(\nu)) d\mu d\nu . \tag{9}$$

Using the properties of the function φ and taking into account the range of the arguments of this function in (9), we find

$$I_2 - I_3 \geq 0 . \tag{10}$$

The formulae (7), (8), (10) prove the lemma.
Theorem.

If $\varphi(x)$ has the properties mentioned in lemma then $R(\ell, x, \xi) \geq 0$ for $0 \leq x \leq \ell \leq \xi \leq L$.
Proof.

Using (6), (3), we find

$$R(\ell, x, \xi) = I_1 + I_2 , \tag{11}$$

where

$$I_k = \iint_{D_k} A(\ell, x, \xi, \eta, \lambda) d\eta d\lambda ; \; k = 1, 2 ;$$

$$D_1 : (0 \leq \eta \leq x, x \leq \lambda \leq \ell), \;\; D_2 : (0 \leq \eta \leq x, 0 \leq \lambda \leq x),$$

$$A(\ell, x, \xi, \eta, \lambda) = g(\eta, x) g(\lambda, \ell) \left[g(\lambda, \xi) g(\eta, \ell) - g(\lambda, \ell) g(\eta, \xi) \right].$$

Using lemma and inequalities $g(\eta, x), g(\lambda, \ell) \geq 0$ which can be easily proved, we obtain that $A(\ell, x, \xi, \eta, \lambda) \geq 0$ in D_1 and hence,

$$I_1 \geq 0 . \tag{12}$$

We rewrite integral I_2 in the form

$$I_2 = \iint_{D_2} B(\ell, x, \xi, \eta, \lambda) d\eta d\lambda ,$$

where

$$B(\ell, x, \xi, \eta, \lambda) = \frac{1}{2} \left[A(\ell, x, \xi, \eta, \lambda) + A(\ell, x, \xi, \lambda, \eta) \right] =$$

$$= \frac{1}{2} \left[g(\lambda, \xi) g(\eta, \ell) - g(\lambda, \ell) g(\eta, \xi) \right] \left[g(\eta, x) g(\lambda, \ell) - g(\eta, \ell) g(\lambda, x) \right].$$

Using lemma, we obtain that $B(\ell, x, \xi, \eta, \lambda) \geq 0$ in D_2 and hence,

$$I_2 \geq 0 . \tag{13}$$

The formulae (11)-(13) prove the theorem.

Conclusions

In the earlier study [1] it has been proved for the plates, which are flat in their natural states, that if the load is localized only in the projecting part of the lower plate then the contact of the plates under the joint cylindrical bending takes place only at one point (except for the fixed point). In the present study the proof is extended to the case the plates are curved in their natural state. The extension of the other results of [1] to this case will be the subject of the further investigations.

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К ТЕОРИИ ИЗГИБА ПРОТЕЗА СТОПЫ, СОДЕРЖАЩЕГО ИСКРИВЛЕННЫЕ ПЛАСТИНЫ

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Доказано, что при совместном цилиндрическом изгибе двух защемленных на одном краю и искривленных в естественном состоянии пластин под нагрузкой, расположенной на выступающей части нижней (большей) пластины, контакт пластин происходит (кроме точки защемления) только в одной точке - на конце верхней (меньшей) пластины. Этот результат может быть использован при исследовании работы каркаса протеза стопы, представляющего собой листовую рессору. Библ. 5.

Ключевые слова: протез стопы, листовая рессора, искривленные пластины, слабый изгиб, область контакта

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