Russian Journal of Biomechanics, Vol. 3, № 3: 23-31, 1999

UNCONFINED COMPRESSION OF THE PERIODONTAL LIGAMENT, INTERVERTEBRAL DISC, ARTICULAR CARTILAGE AND OTHER PERMEABLE DEFORMABLE TISSUES: A POROELASTIC ANALYSIS

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Abstract: In this paper, behavior of the porous tissues such as the periodontal ligament, intervertebral disc, articular cartilage was investigated. They are generally subjected to compressive loads, which are transmitted through the surrounding hard tissues (tooth, bones). To a first approximation, such porous tissues saturated by free fluid may be viewed as poroelastic. The porous medium was considered to be sandwiched between two approached parallel rigid impervious plates. The approximate analytical solution of the problem on stationary flow of the interstitial fluid completely saturating the incompressible porous medium was found. The fields of displacements, strains, stresses of the porous medium and fields of fluid pressure and fluid flow were obtained. The determined solution has qualitatively described the processes proceeding in such porous tissues, in particular, of interstitial fluid movement.

Key words: permeable deformable tissues, poroelastic analysis, unconfined compression, periodontal ligament, periodontal fluid, intervertebral disc, articular cartilage, interstitial fluid

Introduction

In this paper, the behavior of the interstitial fluid that saturates and flows through a deformable porous matrix is considered for the case of unconfined compression of such a porous medium.

The periodontal ligament is a dense connective tissue that surrounds the root of the tooth and attaches it to the alveolar bone. The periodontal ligament behavior is very diverse and depends heavily on the kind of mechanical load acting on the tooth. It was experimentally shown that under a short-term load the periodontal ligament may be considered as a porous material saturated by free fluid with a permeability coefficient k of the order of 10^{-8} $m^2/(Pa{\cdot}sec)$ [18, 20]. Such a load may lead to the different tooth traumas, their kind being mainly determined by the degree of periodontal ligament fluid redistribution [19]. Under a long-term load the periodontal ligament may be viewed as an elastic material [5, 9, 12, 17, 21, 22].

This paper presents the study of such a case of a short-term load, which leads to the translational movement of the tooth. The human periodontal ligament width ranges from 0.1 to 0.3 mm [16, 26]. These values are far less than the tooth root length (the order of 15 mm) [28]. Because of this, the deformation of the porous medium located within a narrow long gap between two approached parallel rigid impervious plates is investigated. The purpose is to determine stresses, strains and displacements of the periodontal ligament solid matrix as well as fluid pressure and fluid velocities.

It should be emphasized that other porous tissues (intervertebral disc, articular cartilage) were viewed as deformable porous solid materials that are saturated by mobile fluid [1, 4, 6-8, 13-15, 23, 24] and the unconfined compression was theoretically investigated in a number of papers [1, 10, 11]. The solution presented in this paper is analytical and describes some processes occurring in the periodontal ligament as well as in articular cartilage, intervertebral disc and other permeable deformable solids.

The problem formulation

It is supposed that the plate size in z-direction is infinite and thus all investigated parameters are independent on z. Then in the stationary process the equations describing the poroelastic behavior of completely saturated porous medium are [2, 3, 20]

$$\begin{cases} -\dfrac{1}{G}\dfrac{\partial p}{\partial x}+\dfrac{\partial^2 u_x}{\partial x^2}+\dfrac{\partial^2 u_x}{\partial y^2}+\dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial x}=0, & (1)\\ -\dfrac{1}{G}\dfrac{\partial p}{\partial y}+\dfrac{\partial^2 u_y}{\partial x^2}+\dfrac{\partial^2 u_y}{\partial y^2}+\dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial y}=0, & (2)\\ \dfrac{\partial^2 p}{\partial x^2}+\dfrac{\partial^2 p}{\partial y^2}=0, & (3)\end{cases}$$

where $u_x(x,y)$ is the x-component of the displacement of the porous medium; $u_y(x,y)$ is the y-component of the displacement of the porous medium; $p(x,y)$ is the fluid pressure; G and ν are the shear modulus and Poisson ratio for the completely saturated porous medium, respectively;

$$e=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y} \text{ is the volume strain of the porous medium.} \quad (4)$$

In this paper, the unconfined compression of two parallel rigid impervious plates is investigated, the points of plates satisfying the conditions $-l\le x\le l$; $y=\pm h$ (Fig. 1). The porous medium is assumed to be located between these plates compressed so that the y-displacement of each plate equals the preassigned magnitude $(\mathbb{R}\tilde{U})$. Thus, the required solutions of equations (1)-(3) are considered to meet the following boundary conditions

$$\begin{aligned} &u_x(x, h)=0 && \text{at } -l\le x\le l;\\ &u_x(x, -h)=0 && \text{at } -l\le x\le l; \qquad (5)\\ &u_y(x, h)=-\tilde{U} && \text{at } -l\le x\le l;\\ &u_y(x, -h)=\tilde{U} && \text{at } -l\le x\le l;\\ &p(\pm l, y)=0 && \text{at } -h\le y\le h. \qquad (6)\end{aligned}$$

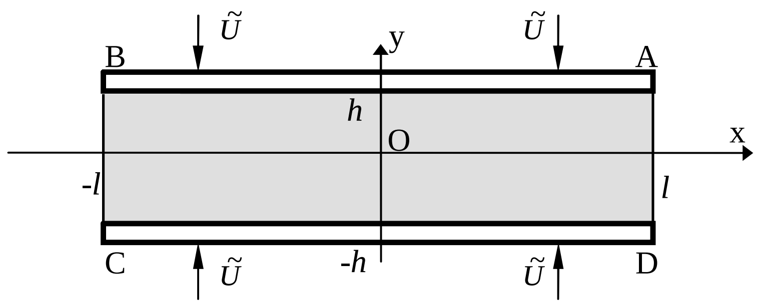

Fig. 1. Unconfined compression of the porous medium (periodontal ligament, articular cartilage, intervertebral disc and others).

It should be noted that the conditions (5) mean the adhesion of solid matrix to the plates and the condition (6) does that the free boundaries ($x=\pm l$) do not inhibit the horizontal fluid exudation.

Determination of the porous medium displacements and fluid pressure

To solve this problem the method in which an approximate analytical solution is built with the help of polynomials is used [27]. One can readily see that the following second-degree polynomial satisfies equation (3)

$$p(x,y) = \frac{A_1}{2}\left(y^2 - x^2\right) + A_2, \tag{7}$$

where A_1 and A_2 are some constants. In this problem, the first-degree terms cannot enter into the polynomial equation because the pressure is independent of the signs of both x and y.

Further, the equations for porous medium displacements will be sought in the form

$$u_x(x,y)=b(y)x, \tag{8}$$

$$u_y(x,y)=a(y). \tag{9}$$

Then using (8) and (9) expression (4) may be written as

$$e=b(y)+a'(y). \tag{10}$$

Substituting (7), (8) and (10) into (1) yields

$$-\frac{1}{G}\left(-A_1 x\right) + b''\left(y\right)x = 0. \tag{11}$$

Hence

$$b''\left(y\right) = -\frac{A_1}{G} \tag{12}$$

and

$$b\left(y\right) = -\frac{A_1}{G}\frac{y^2}{2} + C_1 y + C_2, \tag{13}$$

where C_1 and C_2 are some constants.

Then the x-component of the porous medium displacement may be found by combining equations (8) and (13)

$$u_x\left(x,y\right) = \left(-\frac{A_1}{G}\frac{y^2}{2} + C_1 y + C_2\right)x. \tag{14}$$

The boundary conditions (5) demand

$$\begin{cases} -\dfrac{A_1}{G}\dfrac{h^2}{2} + C_1 h + C_2 = 0, \\ -\dfrac{A_1}{G}\dfrac{h^2}{2} - C_1 h + C_2 = 0, \end{cases} \tag{15}$$

and thus

$$C_1 = 0, \qquad C_2 = \frac{A_1}{G}\frac{h^2}{2}. \tag{16}$$

Then expression (14) takes the following form

$$u_x\left(x,y\right) = \frac{A_1}{2G}x\left(h^2 - y^2\right). \tag{17}$$

Let us determine now $u_y(x,y)$.

Substituting (7), (9), (10), (13), (16) into (2) we derive

$$-\frac{1}{G}A_1 y + a''(y) + \frac{1}{1-2\nu}\left(-\frac{A_1}{G}y + a''(y)\right) = 0 \tag{18}$$

then

$$a''(y) = \frac{A_1}{G}y\,, \tag{19}$$

and therefore

$$a(y) = \frac{A_1}{G}\frac{y^3}{6} + D_1 y + D_2\,, \tag{20}$$

where D_1 and D_2 are some constants.
Thus (9) can be written, in view of equation (20), as

$$u_y(x,y) = \frac{A_1}{G}\frac{y^3}{6} + D_1 y + D_2\,. \tag{21}$$

The boundary conditions (5) demand

$$\begin{cases} \dfrac{A_1}{G}\dfrac{h^3}{6} + D_1 h + D_2 = -\tilde{U}, \\ -\dfrac{A_1}{G}\dfrac{h^3}{6} - D_1 h + D_2 = \tilde{U}, \end{cases} \tag{22}$$

whence it follows that

$$\begin{aligned} D_2 &= 0, \\ D_1 &= -\frac{1}{h}\left(\tilde{U} + \frac{A_1 h^3}{6G}\right), \end{aligned} \tag{23}$$

and then

$$u_y(x,y) = \frac{A_1 y^3}{6G} - \frac{1}{h}\left(\tilde{U} + \frac{A_1 h^3}{6G}\right)y\,. \tag{24}$$

To determine A_2 we assume that the boundary condition on the fluid pressure (6) is satisfied at $y=\pm h$, i.e. at the points A, B, C, D (Fig. 1), then $p(\pm l, \pm h)=0$ and equation (7) yields

$$A_2 = -\frac{A_1}{2}\left(h^2 - l^2\right). \tag{25}$$

Combining equations (7) and (25) we obtain

$$p(x,y) = \frac{A_1}{2}\left(y^2 - h^2 + l^2 - x^2\right). \tag{26}$$

The analysis of (26) shows that the boundary condition (6) is not exactly met. However the error which arises due to approximate problem solving vanishes when $h \ll l$. In fact, for the case of $x=\pm l$ the maximum error appearing by using equation (26), i.e. the maximum difference by absolute value between the fluid pressure obtained using equation (26) and the boundary condition (6), is equal to $p(\pm l, 0)=A_1 h^2/2$. It follows from equation (26) that the maximum pressure proves to equal $p(0,\pm h)= A_1 l^2/2$. Since the relative error is determined by division of these values, then the lower is the ratio h/l, the lower is the relative error.

It is pertinent to note that relations (17), (24) and (26) identically comply with the input equations (1)-(6).

To determine the coefficient A_1 an additional condition is required.

For this purpose let us substitute (17) and (24) into (4), then

$$e = \frac{A_1}{2G}\left(h^2 - y^2\right) + \frac{A_1 y^2}{2G} - \frac{1}{h}\left(\tilde{U} + \frac{A_1 h^3}{6G}\right), \tag{27}$$

and hence the volume strain e of the porous medium is constant with both x and y

$$e = \frac{A_1 h^2}{3G} - \frac{\tilde{U}}{h} = \text{const}(x,y). \tag{28}$$

Further the porous medium is considered to be incompressible, i.e.

$$e=0. \tag{29}$$

The incompressibility condition (29) demands

$$A_1 = \frac{3\tilde{U}G}{h^3} \tag{30}$$

and then equations (17), (24) and (26) are rewritten in the finished form

$$u_x\left(x, y\right) = \frac{3\tilde{U}}{2h^3} x\left(h^2 - y^2\right), \tag{31}$$

$$u_y\left(x, y\right) = \frac{\tilde{U}}{2h^3} y\left(y^2 - 3h^2\right), \tag{32}$$

$$p(x, y) = \frac{3\tilde{U}G}{2h^3}\left(y^2 - h^2 + l^2 - x^2\right). \tag{33}$$

Determination of the fluid velocity vectors and filtered fluid streamlines

To find out the fluid velocity vectors **v** the Darcy's law is used [2, 20, 25]

$$\mathbf{v} = -k\vec{\nabla} p, \tag{34}$$

where k is the permeability coefficient.

In the two-dimensional problem, the components v_x and v_y of the vector **v** (equation (34)) have the form

$$v_x = -k\frac{\partial p}{\partial x}, \tag{35}$$

$$v_y = -k\frac{\partial p}{\partial y}. \tag{36}$$

Then with the help of (33) we find

$$v_x = \frac{3\tilde{U}Gk}{h^3} x, \tag{37}$$

$$v_y = -\frac{3\tilde{U}Gk}{h^3} y. \tag{38}$$

Let us draw now the fluid streamlines. It is known that the streamlines are coincident with the trajectories in the stationary process [29]. In this paper, it is the stationary process which is considered, therefore equations (37) and (38) may be written as

$$\frac{dx}{dt} = \frac{3\tilde{U}Gk}{h^3} x, \tag{39}$$

$$\frac{dy}{dt} = -\frac{3\tilde{U}Gk}{h^3} y. \tag{40}$$

The solutions of (39) and (40) are

$$x = X_\circ e^{\frac{3\tilde{U}Gk}{h^3} t}, \tag{41}$$

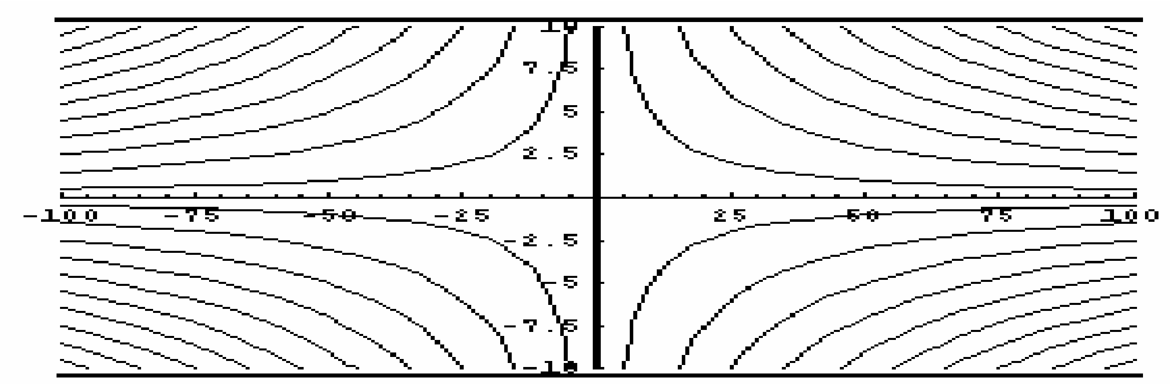

Fig. 2. The streamlines of the filtered fluid.

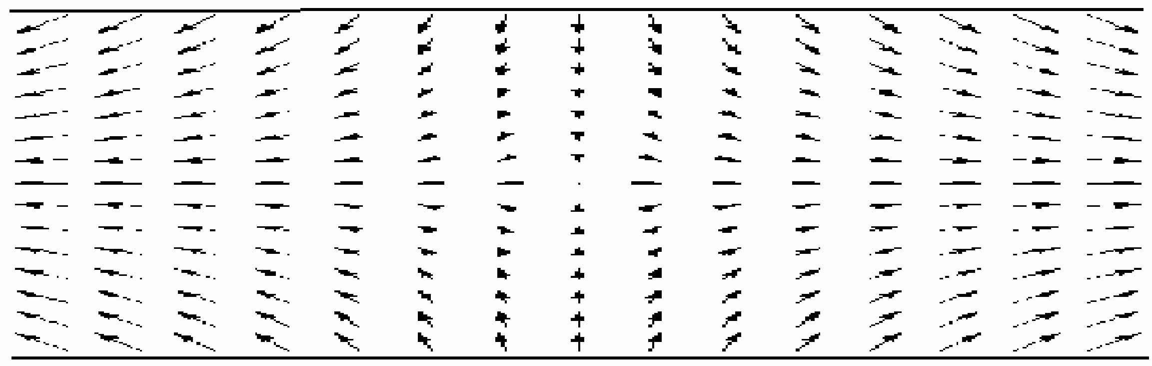
Fig. 3. The vectors of the fluid velocity.

$$y = Y_{\circ} e^{-\frac{3\tilde{U}Gk}{h^3}t}, \tag{42}$$

where t is time; $X_{\circ}$, $Y_{\circ}$ are coordinates of some point at t=0.

Elimination of t from (41) and (42) yields

$$y = \frac{X_{\circ} Y_{\circ}}{x}. \tag{43}$$

Figures 2 and 3 show streamlines of the filtered fluid and vectors of the fluid velocity, respectively (h=10; l=100).

It should be emphasized that a streamline is determined only by the directions of vectors (their magnitudes are of no significance) [29]. That is why if the fluid pressure is known to comply with (7), then relation (43) can be immediately obtained by using (7) instead of (33).

Determination of the strains and stresses of the porous medium

The total stress components of the completely saturated incompressible porous medium are written as [20]

$$\tau_x = 2Ge_x - p, \tag{44}$$

$$\tau_y = 2Ge_y - p, \tag{45}$$

$$\tau_{xy} = G\gamma_{xy}, \tag{46}$$

where the components of the porous medium strains are

$$e_x = \frac{\partial u_x}{\partial x},\ e_y = \frac{\partial u_y}{\partial y},\ \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{47}$$

Combining equations (31)-(33) and (47) we obtain

$$e_x(x,y) = \frac{3\tilde{U}}{2h^3}\left(h^2 - y^2\right), \tag{48}$$

$$e_y(x,y) = \frac{3\tilde{U}}{2h^3}\left(y^2 - h^2\right), \tag{49}$$

$$\gamma_{xy} = -\frac{3\tilde{U}}{2h^3}xy. \tag{50}$$

Then the total stress components of the porous medium are

$$\tau_x(x,y) = \frac{3\tilde{U}G}{2h^3}\left[3\left(h^2 - y^2\right) - l^2 + x^2\right], \tag{51}$$

$$\tau_y(x,y) = \frac{3\tilde{U}G}{2h^3}\left[y^2 - h^2 - l^2 + x^2\right], \tag{52}$$

$$\tau_{xy}(x,y) = -\frac{3\tilde{U}G}{h^3}xy. \tag{53}$$

An additional point to emphasize is that the magnitude F of the force applied to each plate is readily determined as follow

$$F = -2\int_0^l \tau_y(x,\pm h)dx = \frac{2\tilde{U}Gl^3}{h^3}. \tag{54}$$

Conclusions

In this paper, behavior of the porous tissues such as the periodontal ligament, intervertebral disc, articular cartilage was investigated. They are generally subjected to compressive loads, which are transmitted through the surrounding hard tissues (tooth, bones).To a first approximation, such porous tissues saturated by free fluid may be viewed as poroelastic. The porous medium was considered to be sandwiched between two approached parallel rigid impervious plates. The approximate analytical solution of the problem on stationary flow of the interstitial fluid completely saturating the incompressible porous medium was found. The fields of displacements, strains, stresses of the porous medium and fields of fluid pressure and fluid flow were obtained. The determined solution has qualitatively described some processes proceeding in such porous tissues, in particular, of interstitial fluid movement.

Acknowledgements

This work was supported in part by a grant from the INTAS, ?97-32158.

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ДЕФОРМИРОВАНИЕ ПЕРИОДОНТА, МЕЖПОЗВОНКОВОГО ДИСКА, СУСТАВНОГО ХРЯЩА И ДРУГИХ ПРОНИЦАЕМЫХ ДЕФОРМИРУЕМЫХ ТКАНЕЙ: ПОРОУПРУГИЙ АНАЛИЗ

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Ранее было показано (Nyashin M.Y. с соавт., Russian Journal of Biomechanics, 3(1): 89-95, 1999), что периодонт при кратковременном нагружении ведет себя как пористый материал, насыщенный способной перемещаться в нем периодонтальной жидкостью. В настоящей статье приведено решение тестовой задачи, в которой моделировалось движение жидкости в периодонте при поступательном перемещении зуба. Для этого рассматривалась плоская задача о стационарной медленной фильтрации вязкой жидкости, полностью насыщающей деформируемую несжимаемую пористую среду, которая располагается в узкой длинной щели между двумя сближающимися параллельными пластинами. Задача решалась методом полиномов, с помощью которого удалось получить приближенное аналитическое решение задачи. Были определены поля скоростей и линии тока фильтрующейся жидкости, а также получены аналитические выражения для определения всех компонент вектора перемещений точек пористой среды, тензора полных напряжений и тензора малых деформаций пористой среды. Найденное решение качественно совпадает с представлениями о течении периодонтальной жидкости. В статье также отмечается, что предложенная постановка и решение пригодны для исследования поведения некоторых других тканей, которые можно рассматривать как пористые среды. В частности, полученное решение частично описывает сжатие межпозвонкового диска и суставного хряща. Библ. 29.

Ключевые слова: проницаемые деформируемые ткани, пороупругий анализ, периодонт, периодонтальная жидкость, межпозвонковый диск, суставной хрящ, интерстициальная жидкость

Received 11 October 1999